HW: 12.10, 16 (end of page), 22, 39, 42, 43, 48

48

22. from 18 $(f(x) = \sin x)$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

From Taylor's inequality and since $|f^{(n)}(x)| \le 1$ for all n & x,

$$|R_n(x)| \le \frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \text{ for all } x$$

implies the series converges for all x.

39. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\Rightarrow \cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

Convergence: use the ratio test

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right| = \lim_{n\to\infty}\frac{x^2}{n+1} = 0 \text{ for all } x.$$

Thus the radius of convergence is $R = \infty$.

42. $\ln(1+x^2) = \ln(1-(-x^2))$

$$= -\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n} \qquad -1 < -x^2 < 1$$

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$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n} \qquad -1 < x < 1.$$

$$\cancel{\lim_{n\to\infty}\left|\frac{x^{2n+2}}{n+1}\cdot\frac{n}{x^{2n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x|^2 = |x|^2 < 1}$$

~~Thus~~

43. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$$e^{-0.2} \approx 0.81873$$

$$1 + (-0.2) + \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{6} + \frac{(-0.2)^4}{24} + \frac{(-0.2)^5}{120} = 0.81873$$

need 6 terms

48. $\int \frac{e^x - 1}{x}\,dx$

$$= \int\left[\frac{1}{x}\left\{\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \cancel{-1}\right\}\right]dx$$

$$= \int\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!}\right]dx$$

$$= C + x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} + \dots$$

$$= C + \sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$$

16. $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$

$f'''(x) = -2\cdot 3x^{-4}, \dots,$

$f^{(n)}(x) = (-1)^n n!\, x^{-n-1}$

So $f^{(n)}(-3) = (-1)^n n!\,(-3)^{-n-1}$

$$= (-1)^n(-1)^{-n-1} n!\, 3^{-n-1}$$

$$= -n!\, 3^{-n-1}$$

So $$\frac{1}{x} = \sum_{n=0}^{\infty} -\frac{\cancel{n!}\, 3^{-n-1}}{\cancel{n!}}(x+3)^n = -\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}(x+3)^n$$