Math 158-Exam 1

Show all work to receive partial credit!

1. (20 pts) Circle the correct direction field for the given differential equation.

(a) \( \frac{dy}{dx} = 2(y - 1)(y + 1) \)

(b) \( \frac{dy}{dx} = 2(y - 1) \)

(c) In the circled vector fields, plot the following solutions:

(i) the equilibrium solutions;
(ii) the solution through \( y(0) = 0 \);
(iii) the solution through \( y(-1) = -1.5 \);
(iv) the solution through \( y(1) = 1.5 \).
2. (20 pts) Use Euler's Method with step size $h = 0.1$ on the initial value problem

\[
\frac{dy}{dx} = (2y + 3)\sqrt{x}, \quad y(0) = 1,
\]

to approximate the value of $y(0.5)$. Recall that for the differential equation

\[
\frac{dy}{dx} = f(x, y), \quad y(0) = y_0.
\]

Euler's Method has the form

\[
y_{n+1} = y_n + hf(x_n, y_n), \quad x_n = nh, \quad n = 0, 1, 2, \ldots
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0 = 0$</td>
<td>$y_0 = 1$</td>
</tr>
<tr>
<td>$x_1 = 0.1$</td>
<td>$y_1 = y_0 + h(2y_0 + 3)\sqrt{x_0}$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$1 + 0.1(5)\sqrt{0.1}$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$1.1581$</td>
</tr>
<tr>
<td>$x_2 = 0.2$</td>
<td>$y_2 = y_1 + h(2y_1 + 3)\sqrt{x_1}$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$1.1581 + 0.1(2(1.1581) + 3)\sqrt{0.2}$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$1.3959$</td>
</tr>
<tr>
<td>$x_3 = 0.3$</td>
<td>$y_3 = y_2 + h(2y_2 + 3)\sqrt{x_2}$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$1.3959 + 0.1(2(1.3959) + 3)\sqrt{0.3}$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$1.7131$</td>
</tr>
<tr>
<td>$x_4 = 0.4$</td>
<td>$y_4 = y_3 + h(2y_3 + 3)\sqrt{x_3}$</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$1.7131 + 0.1(2(1.7131) + 3)\sqrt{0.4}$</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$2.1196$</td>
</tr>
</tbody>
</table>

\[ y(0.5) \approx 2.1196 \]
3. (a) (17 pts) Use the method of separation of variables to solve the following differential equation with initial condition:

\[ \frac{dy}{dx} = (2y + 3)\sqrt{x}, \quad y(0) = 1. \]

Recall that \( \sqrt{x} = x^{1/2} \).

\[ \int \frac{dy}{2y+3} = \int x^{1/2} \, dx \]

\[ \frac{1}{2} \ln |2y+3| = \frac{2}{3} x^{3/2} + C \]

\[ \ln |2y+3| = \frac{4}{3} x^{3/2} + C \]

\[ 2y + 3 = Ce^{4/3 x^{3/2}} \]

\[ y = Ce^{4/3 x^{3/2}} - \frac{3}{2} \]

\[ 1 = y(0) = C - \frac{3}{2} \implies C = \frac{5}{2} \]

So, \( y(x) = \frac{5}{2} e^{4/3 x^{3/2}} - \frac{3}{2} \).

(b) (3 pts) Using the solution you computed in part (a), compare the actual value of \( y(0.5) \) with the estimate you computed in the previous problem using Euler's Method.

\[ y(0.5) \approx 2.5056 \]

It's a bit bigger.
4. (a) (10 pts) Show that \( y(x) = -1 + 2e^{x^2/2} \) is a solution of
\[
\frac{dy}{dx} = xy + x, \quad y(0) = 1.
\]
\[
\frac{dy}{dx} = 2 \frac{d}{dx} \left[ e^{x^2/2} \right] = 2 \frac{d}{dx} \left[ \frac{x^2}{2} \right] e^{x^2/2} = 2x e^{x^2/2}
\]
\[
-1 + 2e^{x^2/2} + x = -1 + 2x e^{x^2/2} + x = 2x e^{x^2/2}
\]
So \( y \) is a solution and
\[
y(0) = -1 + 2 = 1.
\]

(b) (10 pts) Show that \( y(x) = \frac{x^3}{3} e^{2x} \) is a solution of
\[
\frac{dy}{dx} = 2y + x^2 e^{2x}, \quad y(0) = 0.
\]
\[
\frac{d}{dx} \left[ \frac{x^3}{3} e^{2x} \right] = x e^{2x} + \frac{2}{3} \cdot 2 e^{2x}
\]
\[
2y + x e^{2x} = 2 \cdot \frac{x^3}{3} e^{2x} + x e^{2x}
\]
So \( y \) is a solution and
\[
y(0) = 0.
\]
5. (a) (17 pts) The differential equation model for current $I$ in an RL-circuit has the form

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

If $R = 8$ ohms, $L = 1$ henry, $E = 6$ volts, and $I(0) = 1$ amp, find the equation for the current $I$ at all times $t$. Use the integration factor method to compute the solution, and don’t forget the initial condition.

$$\frac{dI}{dt} + 8I = 6, \quad I(0) = 1$$

4 pts. int factor: $e^{8t}$

$$\int \frac{d}{dt} [Ie^{8t}] dt = \int 6e^{8t} dt$$

7 pts.

$$Ie^{8t} = \frac{6}{8}e^{8t} + C$$

2 pts.

$$I(t) = Ce^{-8t} + \frac{6}{8}$$

1 pt. $1 = I(0) = C + \frac{6}{8} \Rightarrow C = \frac{2}{8} = \frac{1}{4}$

4 pts.

$$\therefore I(t) = \frac{1}{4}e^{-8t} + \frac{3}{4}$$

(b) (3 pts) Using your solution in part (a), calculate the current after 1.5 seconds.

3 pts.

$$I(1.5) = 0.75 \text{ amp}$$
Extra Credit: A tank holds 500 gallons of brine (salt water). Brine containing 2 pounds per gallon of salt flows into the tank at the rate of 5 gallons per minute, and the mixture, kept uniform, flows out at the rate of 10 gallons per minute. If the maximum amount of salt is found in the tank at the end of 20 minutes, what was the initial salt content of the tank?

Points: 5 for the correct model and solution; 5 for the correct computation of the initial salt content.

The other problems are worth more, so don't waste your time on this unless you’re done with the others.

\( x = \text{lbs of salt} \)

\[
\frac{dx}{dt} = 10 - \left(\frac{10}{500-5t}\right)x, \quad x(0) = x_0
\]

\[
\frac{dx}{dt} + \left(\frac{10}{500-5t}\right)x = 10
\]

\[
\text{int factor: } e^{-\int \frac{10}{500-5t} \, dt} = e^{-\frac{1}{5} \ln(500-5t)} = (500-5t)^{-\frac{1}{5}}
\]

\[
\int \frac{d}{dt} \left[ x (500-5t)^2 \right] \, dt = \int 10 (500-5t)^2 \, dt
\]

\[
x(t) = \frac{10}{3} (500-5t)^3 + C
\]

\[
x(t) = -\frac{2}{3} (500-5t) + C (500-5t)^{-2}
\]

\[
x_0 = x(0) = -\frac{1000}{3} + \frac{C}{500^2} \Rightarrow \left(x_0 + \frac{100}{3}\right)500^2 = C
\]

\[
\int_0^{20} x(t) = \frac{2}{3} (500-5t) + \left(250000 x_0 + 25000000\right)(500-5t)^{-2}
\]

\[
x'(t) = \frac{10}{3} + 10 \left(\frac{250000 x_0 + 25000000}{3}\right) (500-5t)^{-3}
\]

\[
= \frac{10}{3} + 25 \times 10^5 \left(\frac{x_0 + 100/3}{3}\right) (500-5t)^{-3}
\]

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\( A: 10-100; B: 80-89; C: 70-79; D: 60-69; F: 59 \downarrow \)}