Visualizations and intuitive reasoning in mathematics

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Abstract. On the basis of historical and didactical examples we consider the role of visualizations and intuitive thinking in mathematics. Examples from the 17th and 19th century have been used as well as smaller empirical studies at upper secondary school level and university level. We emphasize that a mathematical visualization does not reveal its intended meaning. With experience we can learn to interpret the visualization in different ways, depending on what is asked for.

Keywords. Visualization, mathematical concepts, definition.

1. Introduction

The status of visualizations in mathematics has varied from time to time. Mancosu (2005, p. 13) points out that during the 19th century visual thinking fell into disrepute. The reason may have been that mathematical claims that seemed obvious on account of an intuitive and immediate visualization, turned out to be incorrect when new mathematical methods were applied. He exemplifies this with K. Weierstrass' (1815–1897) construction of a continuous but nowhere differentiable function from 1872.[2] Before this discovery, it was not an uncommon belief among mathematicians that a continuous function must be differentiable, except at isolated points. The reason for this was perhaps that mathematicians relied too much on visual thinking. Nevertheless, as the development of visualization techniques in computer science improved in the middle of the 20th century, visual thinking rehabilitated the epistemology of mathematics (Mancosu, 2005, pp. 13–21).

In this paper the role of visualizations and intuitive thinking is discussed on the basis of historical and didactical examples. In a historical study the 17th century debate between the philosopher Thomas Hobbes and the mathematician John Wallis is considered. It seems that one problem was that Hobbes and Wallis were relying a bit too much on visualizations and intuitive thinking instead of formal definitions. Another problem was that at least Hobbes made no clear distinction between mathematical objects and "other objects". We consider

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[2] In 1872 Weierstrass constructed the function

$$f(x) = \sum b^n \cos(a^n x)\pi,$$

where x is a real number, a is odd, $0 < b < 1$ and $ab < 1 + \frac{3\pi}{2}$, which was published by du Bois Reymond (1875, p. 29).

these problems on the basis of two examples that both deal with the connection between the finite and the infinite.

Moreover, we consider some examples from the 19th century, which was a period when mathematical analysis underwent a considerable change. Jahnke (1993) discusses mathematics during the mid 19th century and argues that:

> […] a new attitude in the sciences arose, which led to a sometimes "anti-Kantian" view of mathematics that sought to break the link between mathematics and the intuition of time and space (Jahnke, 1993, p. 265).

In this paper we demonstrate how some fundamental concepts during this time period were defined (or perhaps described) on the basis of a survey written by the Swedish mathematician E.G Björling in 1852. Apparently, one problem was that the definitions of some fundamental concepts were too vague. This caused some problems, for instance the new "rigorous" mathematics contained new types of functions that could be used as counterexamples to some fundamental theorems in mathematical analysis. Another problem was that the definitions were not always generally accepted within the mathematical society.

Furthermore, we discuss some different interpretations of the famous "Cauchy's sum theorem", which was first formulated in 1821. Cauchy claimed that the sum function of a convergent series of real-valued continuous functions was continuous. The validity of Cauchy's sum theorem were frequently discussed by contemporary mathematicians to Cauchy, but there has also been discussions among modern mathematicians regarding what Cauchy really meant with his 1821 theorem. For instance, what did Cauchy mean with his convergence condition, and what did he mean with a function and a variable? In this paper Schmieden and Laugwitz' (1958) non-standard analysis interpretation of Cauchy's sum theorem is considered.

Finally, we consider a didactical issue that deals with Giaquinto's (1994) claim that visual thinking in mathematics can be used to personally "discover" truths in geometry but only in restricted cases in mathematical analysis. In this paper we criticize some of Giaquinto's statements. We stress that it is not proper to distinguish between a "visible mathematics" and a "non-visible" mathematics. Furthermore, we claim that Giaquinto does not take into consideration *what* one wants to visualize and to *whom*.

2. The debate between Hobbes and Wallis

An early example of a discussion on the role of intuition and visual thinking in mathematics is the 17th century debate between the philosopher Thomas Hobbes (1588-1679) and the mathematician John Wallis (1616-1703). Hobbes and Wallis often discussed the relation between the finite and the infinite, or rather, if there is a relation between the finite and the infinite. In this paper two issues of this debate will be considered; *Torricelli's infinitely long solid* and *The angle of contact,* respectively.[3] In connection to the former issue an example

[3] These two examples have also been discussed in (Bråting and Pejlare, 2008).

from a textbook at university level in mathematics will be considered and in connection to the latter issue a didactical study on upper secondary school pupils will be presented.

2.1 Torricelli's infinitely long solid

In 1642 the Italian mathematician Evangelista Torricelli (1608-1647) claimed that it was possible for a solid of infinitely long length to have a finite volume (Mancosu, 1996, p. 130). In modern terminology, if one revolves for instance the function $y = 1/x$ around the x-axis and cuts the obtained solid with a plane parallel to the y-axis, one obtains a solid of infinite length but with finite volume. Sometimes this solid is referred to as *Torricelli's infinitely long solid* (see Figure 1). The techniques behind the determination of Torricelli's infinitely long solid were provided by the Italian mathematician Evangelista Cavalieri's (1598-1647) theory of indivisibles from the 1630:s (Mancosu, 1996, p. 131). However, the difference is that Torricelli was using *curved* indivisibles on solids of *infinitely long* lengths. In (Mancosu, 1996, p. 131) "Torricelli's infinitely long solid" is discussed on the basis of the debate between Hobbes and Wallis.

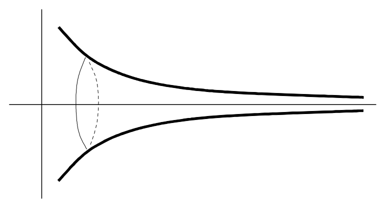

Figure 1: Torricelli's infinitely long solid.

Hobbes rejected the existence of infinite objects, such as "Torricelli's infinitely long solid", since *"[...] we can only have ideas of what we sense or of what we can construct out of ideas so sensed"* (Mancosu, 1996, pp. 145-146). He insisted that every object must exist in the universe and be perceived by "the natural light". Mancosu points out that many 17th century philosophers held that geometry provides us with indisputable knowledge and that all knowledge involves a set of self-evident truths known by "the natural light" (Mancosu, 1996, pp. 137-138). Hobbes stressed that when mathematicians spoke of, for instance, an "infinitely long line" this would be interpreted as a line which could be extended as much as one prefered to. He argued that infinite objects had no material base and therefore could not be perceived by "the natural light". According to Hobbes, it was not possible to speak of an "infinitely long line" as something given. The same thing was valid for solids of infinite length but with finite volume.

Meanwhile, for Wallis, "Torricelli's infinitely long solid" was not a problem as long as it was considered as a *mathematical object*. According to Mancosu, Wallis shared Leibniz' opinion that it was nothing more spectacular about "Torricelli's infinitely long solid" than for instance that the infinite series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

equals 1. If the new method led to the result that infinite solids could have finite volumes, then these solids existed within a mathematical context. Unlike Hobbes, it seems that Wallis (and Leibniz) made a distinction between mathematical objects and "other objects". Perhaps one can also say that Wallis and Leibniz were *generalizing* the volume concept to not only be a measure on finite solids, but also a measure on solids of infinitely long lengths.[4]

A similar (but more modern) problem is to show that the number of elements in the set of all natural numbers is equal to (in terms of cardinality) the number of elements in the set of all positive even numbers. This is done by showing that there exists a "one-to-one" correspondance between the elements of the two sets:

$$\{0, 1, 2, 3, \ldots\}$$
$$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$$
$$\{2, 4, 6, 8, \ldots\}$$

In this case the concept of "number" is generalized. It is relatively easy to determine if the number of elements in two finite sets are equal. One simply has to count the elements in the two respective sets. It is also relatively easy to establish a "one-to-one" correspondance between the elements in that case. However, to determine if the elements in two infinite sets are equal is not that easy. In such a case one has to use a certain *method* to establish a "one-to-one" correspondance between the elements in the two sets. It is important to observe that "Torricelli's infinitely long solid" as well as the example with the comparison between the numbers of elements in two infinite sets are contradictory to "everyday situations" since we obtain "paradoxes". In the latter example the set of positive even numbers is included in the set of natural numbers (although the sets have the same cardinality) and in the former example we obtain a solid with finite volume but infinitely long length.

2.1.1 An example of a concept generalization in a textbook

In the textbook *Calculus – a complete course* (Adams, 2002), which covers the first year of studies in one-dimensional calculus at university level, the volume concept is considered in connection to integration. The textbook introduces that volumes of certain regions can be expressed as definite integrals and thereby determined. Then solids of revolution are considered. For instance, a solid ball can be generated by rotating a half-disk about the diameter of that half-disk. Finally, volumes of solids of infinitely long length are considered in connection to improper integrals. In this case one could perhaps say that the integral concept has been generalized. However, there is no explanation in the textbook that the volume concept actually has been generalized. One of the examples (which is equal to "Torricelli's infinitely long solid") is written in the following way:

The volume of the horn is

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \frac{1}{x^2} dx = \cdots = \pi \; cubic \; units$$

(Adams, 2002, p. 410).

[4]In (Bråting and Öberg, 2005) generalizations of mathematical concepts are discussed in more detail.

One problem of not mentioning that the volume concept has been generalized could be that students believe that we calculate a volume of an "everyday object". Another problem could perhaps be that students think that it is possible to intuitively based on "everyday objects" understand that the volume of the above horn is finite.

2.2 The angle of contact

"The angle of contact", which already occurred in Euclid's *Elements*, appeared to be an angle contained by a curved line (for instance a circle) and the tangent to the same curved line. A dispute between Hobbes and Wallis concerning "the angle of contact" was based on the following two questions:

1. Does there *exist* an angle between a circle and its tangent (see Figure 2)?
2. If such an angle exists, what is the size of it?

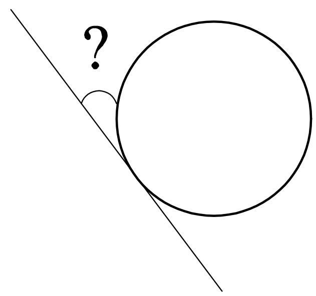

Figure 2: The angle of contact.

Wallis claimed that "the angle of contact" was *nothing*, whereas Hobbes argued that it was not possible that something which could be perceived from a picture could be nothing (Wallis, 1685, p. 71; Hobbes, 1656, pp. 143-144). In fact, this dispute originated from an earlier discussion between Jacques Peletier (1517-1582) and Christopher Clavius (1537-1612) (Peletier, 1563; Clavius, 1607).

According to Hobbes, it was not possible that something that actually could be perceived from a picture drawn on a paper could be nothing. Another reason why "the angle of contact" could not be nothing was the possibility of making proportions in a certain way between different "angles of contact". Hobbes claimed:

> […] an angle of Contingence[5] is a Quantity[6] because wheresoever there is Greater or Less, there is also Quantity (Hobbes, 1656, pp. 143-144).

This statement was perhaps based on Eudoxos' theory of ratios, which is embodied in books V and XII of Euclid's *Elements*. Definitions 3 and 4 of book V states:

> DEFINITION 3. A ratio is a sort of relation in respect of size between two magnitudes of the same kind.
>
> DEFINITION 4. Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another (Heath, 1956, p. 114).

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[5] Hobbes used the term "the angle of Contingence", instead of "the angle of contact".
[6] Hobbes' term "quantity" can be interpreted as "magnitude", which is used in for instance Euclid's *Elements.*

Hobbes, as well as Wallis, discussed the possibility of making proportions between different angles of contact on the basis of a picture similar to Figure 3 below. Hobbes' approach was to compare the "openings" (Hobbes' expression) between two different angles of contact. On the basis of Figure 3, Hobbes claimed that it was obvious that the "opening" between the small circle and the tangent line was greater than the "opening" between the large circle and the tangent line. That is, since one "opening" was greater than the other, the angle of contact must be a quantity since *"wherever there is Greater and Less, there is also quantity"* (see Hobbes' quotation above). From this Hobbes concluded that the angle of contact was a quantity (magnitude), and hence it could not be nothing.

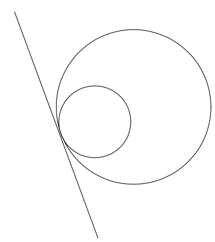

Figure 3: "The angle of contact" in proportion to another angle of contact.

Meanwhile, Wallis (1685) stressed that the "angle of contact" is of "no magnitude". He claimed that *"[...] the angle of contact is to a real angle as 0 is to a number"* (Wallis, 1685, p. 71). That is, according to Wallis it was not possible, by multiplying, to get the "angle of contact" to exceed any real angle (remember Definition 4 above). He pointed out that an "angle of contact" will always be contained in every real angle. However, at the same time he stressed that *"[...] the smaller circle is more crooked than the greater circle"* (Wallis, 1685, p. 91).

Today this is not a problem since we have determined that the answer to the question if the "angle of contact" exists is not dependent on pictures such as Figures 2 and 3 above. Instead the answer depends on which definition of an angle that we are using. In school mathematics an angle is defined as an object that can only be measured between two intersecting segments (Wallin et al., 2000, p. 93). According to such a definition the "angle of contact" is not an angle. However, an angle can be defined differently. For instance, in differential geometry an angle between two intersecting curved lines can be defined as the angle between the two tangents in the intersection point. According to such a definition "the angle of contact" exists and is equal to 0.

2.2.1 Pupils are considering "the angle of contact"

On the basis of a picture similar to Figure 2 above we let 39 pupils of upper secondary school level answer the question:

What is the size of the angle between the circle and its tangent?

According to their teacher these pupils were both motivated and their results in mathematics were above-average. The answers of the pupils were distributed over the following five categories:

1. 9 pupils answered that the angle was 0°.
2. 15 pupils answered that the angle was a fixed value greater than 0°, for instance 45°.
3. 8 pupils answered that the answer depends on where in the picture one measures.
4. 4 pupils answered that the angle does not exist.
5. 3 pupils did not answer.

Several of the 15 pupils in the second category claimed that the angle is 90°. One of the 8 pupils in the third category was formulated in the following way:

> It depends on where one measures. Since the circle is curved the angle gets greater and greater for every point. At the point where the circle and the tangent meets the angle is 0°.

Another pupil in the third category answered:

> Exactly in the tangent point the angle is 0°.

It seems that the pupils' approach was to carefully study the picture to find the answer. Roughly speaking, they were trying to find the answer "in the picture". Apparently, most of the pupils did not base their answers on a formal definition.

One of the 4 pupils in the fourth category gave the following answer:

> It is not an angle since the circle is round.

Of course one cannot be certain that this answer was based on the definition of angle. But perhaps the pupil understood that something was wrong but could not explain why. Perhaps the task was different than the pupil was used to, for instance, normally curved lines have nothing to do with angles. Another pupil in the same category answered:

> Does a circle really have an angle? If it has, it has infinitely many angles.

This pupil does not refer to the definition of angle either, but similar to the answers in this category just mentioned, it seems that this pupil also understood that something was not as it used to be. The two remaining pupils in this category did however refer to the definition of angle.

One possible reason why most of the pupils did not use the definition of an angle could be that they are not used to apply definitions. Perhaps the problems in their textbooks are not based on using formal definitions. Nevertheless, the historical debate concerning the existence of "the angle of contact" could be one way to demonstrate the need of formal definitions in mathematics.

3. Some examples of mathematical analysis from the mid-19th century

In the history of mathematics the 19th century is often considered as a period when mathematical analysis underwent a major change. There was an increasing concern for the lack of "rigor" in analysis concerning basic concepts, such as functions, derivatives, and real numbers (Katz, 1998, 704-705). For instance, the mathematicians wanted to loose the connection between mathematical analysis and geometry. The definitions of several fundamental concepts in analysis were vague and gave rise to different views of not only the definitions, but also of the theorems involving these concepts. Furthermore, mathematicians did not always use the same definition of fundamental concepts. Disputes regarding the meaning of fundamental concepts and the validity of some theorems started to arise. Two reasons to this may have been vague definitions and the lack of generally accepted definitions of fundamental concepts in mathematics.

Jahnke (1993) discusses a new emerging attitude among mathematicians during the mid 19th century whose aim was to erase the link between mathematics and the intuition of time and space. He argues that in the natural sciences in general the ambition of requiring new empirical knowledge became less important, instead, science should focus on the "understanding of nature and culture" (Jahnke, 1993, p. 267). Furthermore, Jahnke states:

> Rather than considering pure mathematics in terms of algorithmic procedures for calculating certain magnitudes, the emphasis fell on *understanding* certain relations from their own presuppositions in a purely conceptual way. To understand given relations in and of themselves one must generalize them and see them abstractly (Jahnke, 1993, p. 267).

Laugwitz (1999) considers the 19th century as a ”turning point" in the ontology as well as the method of mathematics. He argues that instead of using mathematics as a tool for computations, the emphasis fell on conceptual thinking. Laugwitz continues:

> The supreme mastery of computational transformations by Gauss, Jacobi and Kummer was beyond doubt but had reached its practical limits (Laugwitz, 1999, p. 303).

In Sections 3.1 and 3.2 of this paper we consider some examples of how fundamental concepts in analysis were defined (or perhaps described) during the mid 19th century. We also discuss which effect vague definitions of fundamental mathematical concepts can have on theorems in analysis. The examples are based on the Swedish mathematician E.G. Björling (1808-1872), who was an associated professor in Sweden during this time period. In Section 3.1 Björling's view of some fundamental concepts in analysis will be considered. In Section 3.2 we will consider the famous "Cauchy's sum theorem", which was first formulated in 1821. In particular, we will focus on some different interpretations of what Cauchy really meant with his theorem. We will consider modern interpretations of Cauchy's sum theorem as well as interpretations of some contemporary mathematicians to Cauchy.

3.1 E.G. Björling's view of fundamental concepts in mathematical analysis

Björling lived during the above mentioned time period when mathematics underwent a considerable change. One can discern the "old" mathematical approach as well as the "new" mathematical approach in Björling's work. For instance, Björling had an "old-fashioned" way of considering functions when he sometimes considered them as something that already existed and the definition worked as a description. At the same time, Björling tried to develop new concepts in mathematics. A closer look of some of Björling's work will be considered.

In a paper from 1852 Björling included a survey where he defined (or rather described) fundamental concepts in mathematical analysis; for instance functions, derivatives and continuity. Although the purpose with the paper was to consider Cauchy's sufficient condition for expanding a complex valued function in a power series[7], Björling claimed that it was necessary to clarify some fundamental concepts in both real and complex analysis. According to Björling there seemed to exist different views of some of the most fundamental concepts in analysis. He stated:

> It goes without saying, that it has been necessary to return to some of the fundamental concepts in higher analysis, whose conception one has not yet generally agreed on […] It was, from my point of view, necessary, that I in advance – and before the main issue was considered – gave *my own* conception of these fundamental concepts and of these propositions' general applicability, then not only the base, which I have built, would be properly known, but also every misunderstanding of the formulation of the definite result would be prevented (Björling, 1852, p. 171).

Björling begins his survey by considering the function concept. He describes a function as

> […] an analytical expression which contains a real variable *x* (Björling, 1852, p. 171).

Björling certainly defined functions, but sometimes he seemed to consider the definition of a function as a description of something that already exists. As a consequence of his definition of a function, Björling considered every variable expression as a function . Of course this differs form the modern function concept in several ways. For instance, for Björling a function did not need to be single-valued (which will be exemplified below). One possible way to interpret Björling's function concept is as a rule which "tells you what to do with the variable x".

Let us consider three functions that Björling discussed in his survey from 1852. In modern terminology these three functions would be expressed as;

$$f(x) = \frac{x}{|x|}\,^{8}, \qquad g(x) = \frac{\sqrt{x}-\sqrt{a}}{x-a} \quad and \quad h(x) = |x|.$$

These functions are graphically represented in Figure 4 below.

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[7] Björling's view of Cauchy's theorem on power series expansions of complex valued functions are discussed in (Bråting, 2010+).

[8] Björling used the notation $\frac{x}{\sqrt{x^2}}$.

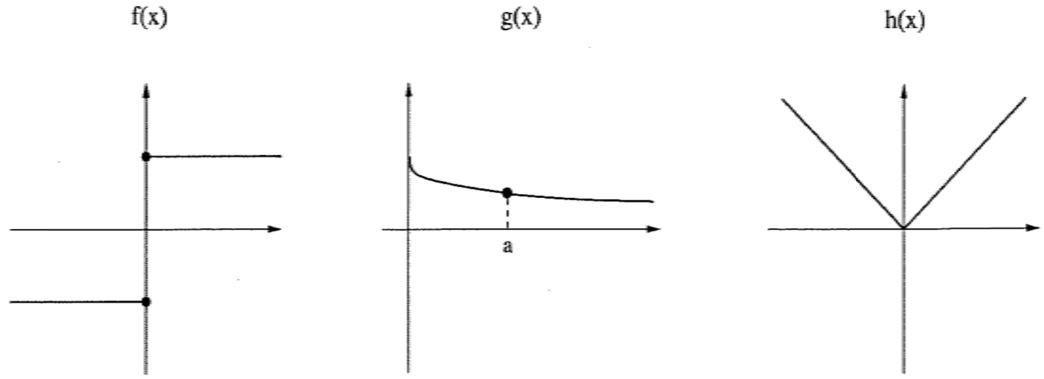

Figure 4: A graphical representation of how Björling may have considered the functions *f, g,* and *h*.

Björling considered $f(x)$ as a function which attains the *two* values ± 1 at $x = 0$. However, it had no derivative at $x = 0$ since the function representing the derivative jumps at $x = 0$. One problem of allowing functions to attain more than one value is that it becomes difficult to conclude what limit function a certain sequence of functions converges to (which is discussed further in Section 3.2 in this paper). Perhaps one can say that we have made it easier for us today since a function is always defined on a specific domain. In the above example this would imply that one need not take into account which value(s) the function $f(x)$ obtains at $x = 0$ since it is not defined at this point.

According to Björling, the function $g(x)$ attains the one value $\frac{1}{2\sqrt{a}}$ at $x = a$, since

$$\lim_{\Delta \to 0} g(a + \Delta) = \frac{1}{2\sqrt{a}}.$$

In modern terminology we would say that Björling considers $g(a)$ as a removable discontinuity. Björling did not consider whether the derivative of $g(x)$ at $x = a$ existed or not, but probably (on the basis of similar examples in Björling's 1852 survey) Björling would have said that the derivative at $x = a$ is equal to $-\frac{1}{8a\sqrt{a}}$ since

$$\lim_{\Delta \to 0} \frac{g(a + \Delta) - g(a)}{\Delta} = -\frac{1}{8a\sqrt{a}}.$$

According to Björling, the function $h(x)$ attains 0 at $x = 0$ and the derivative at $x = 0$ exists and is equal to the two values ± 1. In fact, Björling (1852) claimed that *"[...] generally, the derivative of a function $f(x)$, at a specific point x_0, can only obtain a finite and determined quantity" if $f(x_0)$ is single-valued*" (Björling, 1852, p. 177).

On the basis of these three examples, it seems that Björling's approach was to investigate the behavior of mathematical objects on the account of their "natural properties". At least one gets the impression that, for Björling, it was already presupposed that the expression written on the paper was a function and the task for Björling was to discover its exact properties. Perhaps one can say that Björling tried to "find answers in the graphs of the functions".

3.2 Cauchy's sum theorem

In 1821 the French mathematician A.L Cauchy (1789-1857) claimed that the sum function of a convergent series of real-valued continuous functions was continuous. Cauchy's proof of the theorem was relatively concise and imprecise, which led to different interpretations of Cauchy's formulation of the theorem. Some contemporary mathematicians to Cauchy criticized the validity of the theorem and came up with exceptions as well as corrections of the theorem. Furthermore, the formulation of Cauchy's 1821 theorem has been frequently discussed among mathematicians and historians of mathematics in modern time. For instance, if one interprets Cauchy's convergence condition with, in modern terminology, *pointwise convergence* Cauchy's 1821 theorem is wrong. Meanwhile, if one interprets Cauchy's convergence condition with the modern concept *uniform convergence*, Cauchy's 1821 theorem is correct.

In 1826 the Norwegian mathematician N.H Abel (1802-1829) came up with exceptions to Cauchy's sum theorem. Abel had constructed new types of functions that were sums of trigonometric series, which Cauchy probably had not anticipated when he formulated the theorem in 1821. It turned out that some of these functions could be used as counterexamples to Cauchy's sum theorem. For instance, in (Abel, 1826, p. 316) Abel emphasized that the trigonometric series

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots$$

was an exception to Cauchy's theorem. Apparently, although this series is a convergent series of real valued continuous functions, the sum is discontinuous at $x = 2k\pi$, for each integer *k* (see Figure 5).

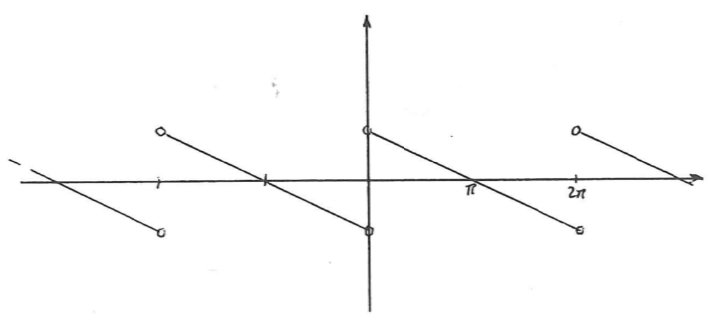

Figure 5: A (modern) graphical representation of the sum of the series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots$

However, it was not only the new functions that led to problems regarding the validity of Cauchy's sum theorem. It seems that the mathematical theory had reached a point where the convergence condition was not precise enough to exclude counterexamples such as Abel's. It turned out that a series of functions can converge in different ways and it was therefore necessary to specify the convergence condition in Cauchy's theorem. In fact, during this time period there were already attempts to distinguish between different convergence concepts. For instance, Björling (1846) tried to explain and prove Cauchy's sum theorem on the basis of his own distinction between "convergence for every value of *x*" and "convergence for every *given* value of *x*", where the former notion was the stronger convergence condition.[9] Perhaps "convergence for every value of *x*" in connection with Cauchy's sum theorem was an attempt to express what in modern terminology could be described as

$$\sup_x \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{\infty} f_k(x) \right| \to 0$$

when $n \to \infty$, that is uniform convergence of the partial sums. However, one problem for Björling was that he did not have a proper way of connecting the variables *n* and *x*. As Grattan-Guinness (2000) points out

> […] during the 19th century there was a problem to distinguish between the expressions "for all x there is a y such that…" and "there is a y such that for all x…" (Grattan-Guinness, 2000, p. 70).

Later both Stokes (1847) and Seidel (1848) came up with corrections to the theorem, but it was not until 1853 that Cauchy modified his 1821 theorem by adding the stronger convergence condition "always convergent" to his 1821 version. In an example Cauchy clarifies that if an equality holds "always" it must hold for $x = 1/n$, that is, he allows *x* to depend on *n*.

3.2.1 A modern view of Cauchy's sum theorem

A standard theorem in modern analysis is the following: If a sequence of real valued continuous functions, f_n, converges uniformly to a function *f*, then *f* is a continuous function. In this case uniform convergence means that the maximum value of $|f_n(x) - f(x)| \to 0$ when $n \to \infty$. That is, for each *n* we choose the "worst" *x*, which makes $|f_n(x) - f(x)|$ as large as possible. If this absolute value still "tends to 0" while "*n* tends to infinity" then f_n converges uniformly to *f*. One obtains Cauchy's sum theorem for

$$f_n(x) = \sum_{1}^{n} g_k(x),$$

where g_k are continuous functions.

The non-standard analysis interpreters' Schmieden and Laugwitz (1958) claim that Cauchy's 1821 theorem was correct, at least if one uses their own theory based on infinitesimals. However, Schmieden and Laugwitz' interpretation has been discussed among historians of mathematics. The issue has been to interpret what Cauchy meant with his expression (mentioned above)

$$x = \frac{1}{n}.$$

Cauchy defined an infinitesimal as a variable which becomes zero in the limit (Cauchy, 1821, p. 19). However, this definition has been interpreted in different ways. For instance, Giusti (1984) argues that $x = \frac{1}{n}$ should *"[…] be seen as an ordinary sequence having 0 as a limit"*

[9] Björling's convergence concepts are considered in detail in (Bråting, 2007).

(Giusti, 1984, pp. 49-53). In modern terminology this is often written as $x_n = \frac{1}{n}$. Meanwhile, Laugwitz (1980) claims that *"[...] for Cauchy a real number can be decomposed into two parts $x + \alpha$, where x is the standard part and α an infinitesimal quantity"* (Laugwitz, 1980, p. 26). The infinitesimal α can be generated by a sequence having zero as a limit, for instance $\alpha = 1/n$. If one uses Laugwitz' interpretation it appears that Cauchy's 1821 theorem was correct. In fact, by using Laugwitz' infinitesimal theory one can even show that Cauchy's convergence condition from 1821 was weaker than uniform convergence, yet sufficient for the theorem to be true (Palmgren, 2007, pp. 171-172). However, in (Bråting, 2007, p. 534) it is argued that Laugwitz' theory presupposes the modern function concept, which was not available for Cauchy. Remember from Section 3.1 above that Björling, who was a contemporary mathematician to Cauchy, considered a function as *"[...] an analytical expression which contains a variable x"*. Bråting (2009) claims that

> [...] with such an imprecise way of defining functions it seems unlikely that a weaker convergence concept than uniform convergence can guarantee continuity in the limit (Bråting, 2009, p. 18).

Let us consider an example of a sequence of functions which shows the difficulty of deciding what will happen in the limit with an imprecise function concept and without the strong condition uniform convergence. Consider the sequence of functions

$$h_n(x) = \begin{cases} \sin 2\pi nx\,, & x \in [0, 1/n] \\ 0\,, & elsewhere \end{cases}$$

The successive graphs of the functions h_1, h_2, h_4 and h_8 are illustrated in Figure 6 below.

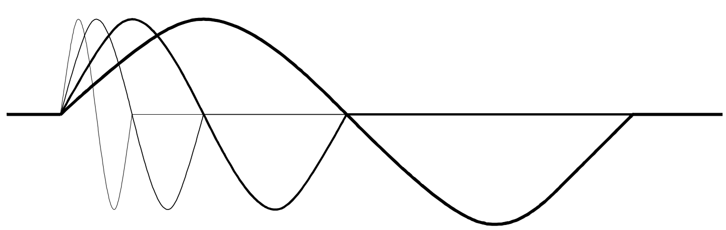

Figure 6: The successive graphs of the functions h_1, h_2, h_4 and h_8.

In modern terminology the sequence of functions h_n converges pointwise to 0 and the 0-function is continuous. However, h_n does not converge uniformly since

$$h_n\left(\frac{1}{4n}\right) = 1$$

for each *n*. Hence, the sequence of functions h_n does not converge uniformly, but yet the limit function is continuous. It is nothing wrong with this since uniform convergence only is a sufficient condition in the modern version of Cauchy's sum theorem.

Without the modern function concept it would be difficult to conclude what happens near 0. One should have in mind that the concept "limit function" is based on pointwise convergence and in this case a clear distinction between $x = 0$ and $x > 0$. In modern terminology we argue that h_n converges pointwise to 0 (and therefore 0 is the limit function) since for each fix $x > 0$ we get $h_n(x) = 0$ if we choose *n* sufficiently large and $h_n(0) = 0$ for each *n*.

However, it is necessary to explain exactly what is meant with ”what happens when n grows?” Otherwise, one can perhaps obtain the following answer to what happens with the graphs of h_n:

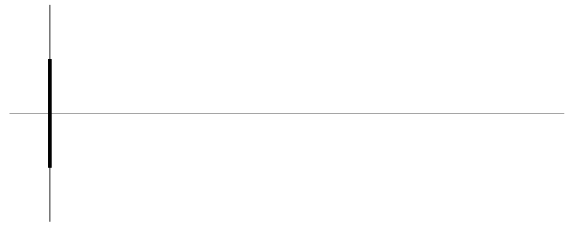

Figure 7: One possible answer to “what happens with the graphs of h_n when n grows?”

There is nothing wrong with this answer if we choose another convergence concept, for instance “pointwise convergence in two dimensions” (that is of a graph). Remember from Section 3.1 above that Björling considered ”multi-valued” functions, for instance Björling claimed that $y = \frac{x}{|x|}$ obtained the two values ± 1 at $x = 0$.

In fact, we let twenty university students at the course ”One-dimensional analysis” answer the question ”What happens at 0 when n turns to infinity?” on the basis of Figure 6 above. Eighteen of the twenty students gave the answer illustrated in Figure 7. One reason for this may be that one wants to believe that there must be something left when “the sine wave” get compressed? Or perhaps the students’ answers were based on some physical argumentation.

In the above example we have interpreted the expression $\frac{1}{4n}$ as an ordinary sequence and not as an infinitesimal quantity. However, if the expression $\frac{1}{4n}$ is interpreted as an infinitesimal quantity it can be shown, by using non-standard analysis, that the sequence of functions h_n satiesfies Cauchy’s sum theorem.[10]

This example shows that there are limits to what a visualization in mathematics can achieve. In the first place, one has to decide what convergence condition should be used, which requires a formal definition. Then the answer can be uniquely determined. Apparently, although one knows the formal definition of pointwise convergence it can be difficult to conclude that the limit function in the above example becomes 0 (remember the students’ answers that were mentioned above).

4. What can visualizations achieve?

A closely related issue to Björling’s view of mathematical concepts comes up in Giaquinto’s (1994) claim that visual thinking can be a means of discovery in geometry but only in severely restricted cases in analysis. By discoveries he does not mean scientific discoveries, but how one personally realizes that something is true. He suggests that visualizing becomes unreliable whenever it is used to discover the existence or nature of the limit of some infinite processes. One gets the impression that Giaquinto tries to distinguish between a “visible

[10] This is demonstrated in detail in (Bråting, 2007, pp. 531-532).

mathematics" and a "less visible mathematics". However, to divide mathematics in such a way can be interpreted as if one part of mathematics is more directly connected to empirical reality and the other part is abstract. Giaquinto stresses;

> […] geometric concepts are idealizations of concepts, with physical instances, meanwhile the basic concepts of analysis are non-visual since they are not equivalent to any (known) concepts which can be conveyed visually (Giaquinto, 1994, p. 804).

It seems that Giaquinto does not take into account *what* one wants to visualize and to *whom*. The main problem is that Giaquinto does not seem to consider *who* is supposed to make the discovery.

In order to investigate Giaquinto's claims, we will consider one of his own examples.[11] With this example Giaquinto wants to show that visualizing the limit of an infinite process sometimes can be deceptive. He considers the following sequence of curves: the first curve is a semicircle on a segment of length *d*; dividing the segment into equal halves the second curve is formed from the semicircle over the left half and the semicircle under the right half; if a curve consists of 2^n semicircles, the next curve results from dividing the original segment into 2^{n+1} equal parts and forming the semicircles on each of these parts, alternatively over and under the line segment; see Figure 8. We can notice that at those points where two of the semicircles touch each other, we will have singularities. The smaller the semicircles are the more singularities we get.

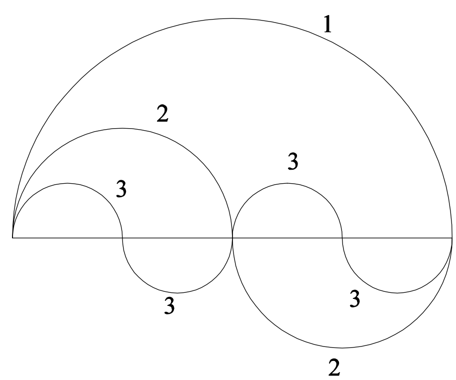

Figure 8: The sequence with semicircle curves.

Giaquinto claims that since the curves converges to the line segment, the limit of the lengths of the curves appears to be the length of this segment. He points out that this belief is wrong, since the sequence of the lengths of the curves will be

$$\frac{\pi d}{2}, \frac{\pi d}{2}, \frac{\pi d}{2}, \frac{\pi d}{2}, \ldots$$

and therefore converge to

$$\frac{\pi d}{2}.$$

[11] The investigation of this example is also considered in (Bråting and Pejlare, 2008).

Furthermore, Giaquinto argues that this example lends credence to the idea that visualizing is not reliable when used to discover the nature of the limit of an infinite process.

However, it seems that Giaquinto does not take into consideration that the interpretation of a visualization does not necessarily have to be unique. We could for example in this case consider the limit of the lengths of the curves, or we could consider the length of the limit function. Depending on how we interpret the visualization we get different results. If we look at the lengths of the curves and take the limit we get the result $\frac{\pi d}{2}$. But, if we instead consider the length of the limit function, then the result is the length of the diameter, that is the result is d. Thus, depending on what question we want to answer we have to interpret the visualization in different ways.

In the visualization of the semicircles much is left unsaid. The visualization does for instance not tell us to look for the limit of the lengths of the curves or for the length of the limit function. We believe that our mathematical experience, as well as the context, is important while interpreting the visualization and "seeing" the relation. Giaquinto does not seem to take into account that people are on different levels of mathematical knowledge, and that visualizations can certainly be sufficient for convincing oneself of the truth of a statement in mathematics, if one has sufficient knowledge of what they represent. A person with little mathematical experience may not realize that the visualization can be interpreted in more than one way, giving different results. With experience we can learn to interpret the visualization in different ways, depending on what is asked for. The more familiar we become with mathematics the more we may be able to "read into" the visualization.

Apparently Giaquinto believes that there exists a definite visualization which reveals its meaning to the individual. However, we do not believe that it is quite that simple. We argue that the individual interacts with a mathematical visualization in a way which is better or worse depending on previous knowledge and on the context. This interaction is important and may even be necessary; the meaning of the visualization is not independent of the observer. A concrete example is when a teacher illustrates a circle by drawing it on the blackboard, as in Figure 9 below.

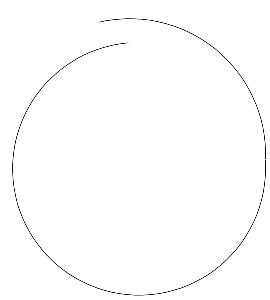

Figure 9: "The circle on the blackboard".

The picture on the blackboard is not a circle, since it is impossible to draw a perfect circle. But for a person who knows that a circle is a set of points in the plane that are equidistant from the midpoint, the picture on the blackboard is sufficient to understand that the teacher is talking about a “mathematical” circle. However, for a child who has never heard of a circle before, the figure of the blackboard probably means something else. By looking at the circle in Figure 9 the child may even think that a circle is a ring which is not connected at the top. The point is that visualizations can certainly be sufficient for convincing oneself of the truth of a statement in mathematics, provided that one has sufficient knowledge of what they represent.

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