Practical Rationality, the Disciplinary Obligation, and Authentic Mathematical Work: A Look at Geometry

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Grossman and McDonald (2008) recently argued that the research community needs to move its "attention beyond the cognitive demands of teaching … to an expanded view of teaching that focuses on teaching as a practice (p. 185)." Building on the work of Bourdieu (Bourdieu and Wacquent, 1992; Bourdieu, 1985, 1998), Herbst and Chazan (2003, 2006) have written about mathematics teaching as a practice, just as law and medicine are considered practices, in an attempt to better understand the rationality that produces, regulates, and sustains mathematics instruction. This *practical rationality* is the commonly held system of dispositions or the "feel for the game" (Bourdieu, 1998, p. 25) that influences practitioners as to those actions that are appropriate in the classroom.

It is practical rationality that:

> not only enables practices to reproduce themselves over time as the people who are the practitioners change, but also regulates how

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> instances of the practice are produced and what makes them count as instances. (Herbst and Chazan, 2003, p. 2)

To better understand the practice of mathematics teaching, whether to improve it or communicate it to others, one must understand the practical rationality that guides it. However, practical rationality often "erases its own tracks" (Herbst and Chazan, 2003, p. 2) so that its practitioners come to view these practices as being natural. This rationality provides the regulatory framework that socializes its current and future practitioners into ways of thinking and acting that conform to expectations. For that reason, it is important to bring to the forefront a deliberate, conscious understanding of the rationality that drives the practice of mathematics teaching.

While practical rationality allows for a certain amount of diversity in its similarity, it is nevertheless given structure and cohesion by a complex system of norms. The word "norms" is used here not in the sense of a "standard" or something that is necessarily desirable, nor in the sense of an absolute requirement, but rather to denote that which is customary, typical, commonplace — behavior that passes without remark. Departures from a norm may occur, but when they do they are usually remarked upon and justified, thereby simultaneously confirming the norm and articulating the conditions under which it may be breached. These norms, and the grounds to which practitioners appeal to justify the norms and their breaches, provide the persistent continuity of the practice.

Although norms are held in common among practitioners, they are usually not explicitly taught to novices. On the contrary, well before future teachers ever enroll in education courses, they already have firmly-established ideas about schools in general and mathematics instruction in particular (Ball, 1988a, 1988b). Through an apprenticeship of observation, they develop deep-seated ideas about mathematics and its teaching and learning (Lortie, 1975). These ideas often form the foundation on which they will eventually build their own practice of mathematics teaching (Millsaps, 2000; Skott, 2001).

A Look at Geometry

What do we know about the rationality that underpins geometry instruction? Herbst and Brach (2006) draw our attention to the practice of geometry instruction and provoke thought regarding the norms surrounding the teaching of proof,[2] but what about other key components of geometry courses? For example, definitions play a critical role in geometry. What norms exist for the teaching of definitions in geometry? Is the norm for students to be presented with finalized definitions? Under what conditions are students given opportunities to create, reflect on, and compare definitions (de Villiers, 1998)?

What is normative in regards to the introduction and use of the diagrammatic register (Weiss & Herbst, 2007) commonly encountered in geometry classes?

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[2] Additional information on norms surround proving and proof is found at Herbst and Brach (2006).

What rationality guides teachers' and students' expectations in regard to the role of perception in the reading of geometric diagrams? What norms influence the teaching of subtle, yet key, concepts of geometry like existence and uniqueness? Are students given impossible problems[3] as a means to discover existence? Are students allowed to explore situations that demonstrate uniqueness?[4]

Mathematics: Teachers' Beliefs and Practices

While many of the above questions are particular to geometry, others apply to the many branches of mathematics. Is it normative to encourage students to modify a problem (either to make it tractable, or to generate new avenues for exploration), or to introduce their own assumptions when solving problems? Do teachers commonly encourage students to pose their own problems? Do teachers model or introduce strategies like Brown and Walters' (2004) "what-if-not" strategy as a relatively simple means of generating new problems in their teaching practice?[5]

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[3] Questions of existence (or non-existence) arise in a wide range of problems, such as: Can one form a triangle with sides of lengths 2 cm, 3 cm and 10 cm? Can one locate a point in the interior of any polygon that is equidistant from all of its vertices? Under what conditions can a circle be constructed tangent to two intersecting lines at two specified points? This last problem is shown as a part of an instructional episode modeled in the ThEMaT (Thought Experiments in Mathematics Teaching) animations found at http:grip.umich.edu/themat.

[4] Questions of uniqueness in geometry likewise arise in a range of problems, such as: Given two sides of a triangle and a non-included angle, how many different triangles can be constructed? Given any parallelogram, is there a uniquely determined quadrilateral whose midpoints are the vertices of the given parallelogram?

[5] For example of a what-if-not application, consider how a compass and straightedge are used to construct a perpendicular bisector for a given line segment. Applying the "what-if-not" strategy could lead to the following questions. What if you wanted to construct a bisector that was not perpendicular to the line segment? How could you construct a perpendicular that did not bisect the segment?

Unfortunately, a large number of teachers view mathematics "as a discipline with *a priori* rules and procedures that … students have to learn by rote" (Handal, 2003, p. 54). For many teachers in the U.S. "knowing" mathematics is taken to mean being efficient and skillful in performing rule-bound procedures and manipulating symbols (Thompson, 1992). Ball (1988b), in her doctoral study of preservice teachers' ideas about the sources of mathematics and how mathematics is justified, found that many of them viewed mathematics as a mostly arbitrary collection of facts. While there are surely many factors that influence teachers' practices, it would be naïve to assume that these and other beliefs teachers hold do not play a significant role. As a consequence, mathematics students often are "not expected to develop mathematical meanings and they are not expected to use meanings in their thinking" (Thompson, 2008, p. 45).

Targeting the Disciplinary Obligation

Herbst and Balacheff (2009) have suggested four obligations of teachers that frame their practical rationality. These obligations — which they refer to as the *disciplinary*, *individual*, *interpersonal*, and *institutional* obligations — may be invoked by teachers to justify normal instruction, but they also have the potential to organize a departure from normative practice.

Of the four, we focus here on the disciplinary obligation — the obligation of the teacher to faithfully represent the discipline of mathematics. We begin from the premise that if teachers come to a more textured and authentic view of mathematics, this could lead to changes in what teachers deem as valid representations of mathematics, in the mathematical tasks they assign students, and in the ideas and attitudes they foster in students. Following Yackel and Cobb (1996) we note that what is taken as

> mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity. (p. 460)

This focus on the disciplinary obligation brings into focus the question of what kind of work is "legitimized as acceptable mathematical activity" (in the words of Yackel and Cobb)? How does it correspond to the kind of work that mathematicians do?

Authentic Mathematical Practices

In Weiss, Herbst and Chen (2009) it was noted that, while the notion of "authentic mathematics" is frequently invoked in the literature, nevertheless "many of those who call for 'authentic mathematics' (or who use similar words or phrases, such as 'genuine' or 'real') in the classroom are actually talking about different things" (p. 276). In particular, Weiss, Herbst and Chen identify four distinct meanings of the slogan "authentic mathematics education". Of particular interest to us here is the one they refer to as AM_P, i.e. the call for the cultivation of the *practices* that characterize the work of research mathematicians. Note, however, that in acknowledging the polysemy of the phrase "authentic mathematics" we allow for, and even anticipate, the possibility that these multiple kinds of "authenticity" may come into conflict with one another.

Mathematicians, those whose goals are to generate new and refine existing mathematical ideas and methods, are more than just proficient at mathematics. While they demonstrate exactly those qualities and competencies that have been identified by the National Research Council (2001) as goals of mathematics learning (namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition), mathematicians also demonstrate habits of "mathematical wondering" and an appreciation of mathematics that extends past their professional careers into their personal lives. They spend much of their time crafting new problems from existing ones, both out of pragmatism (some problems are more tractable than others at a given time) and out of curiosity.

In seeking to articulate the elements of the sensibility that characterizes mathematicians' practices, Weiss (2009) analyzed a collection of narratives written by and about research mathematicians. This analysis reveals the fundamentally generative nature of mathematical practice, in which *problem posing* (asking fruitful and difficult questions of oneself and others) plays a role just as important as *problem solving*. The result of Weiss' analysis is a partial model of the mathematical sensibility, consisting of 15 mathematical dispositions, organized in 8 dialectical pairs (one disposition is its own dialectical counterpart). Weiss refers to the first five of those dispositions as *generative moves* by which a problem currently under consideration (whether solved or unsolved) can spawn a number of related problems. The five generative moves are shown in Fig. 1.

(strengthen/weaken) hypothesis	⟷	(strengthen/weaken) conclusion
generalize	⟷	specialize
consider converse	⟷	consider converse

Figure 1. Generative moves for problem posing taken from Weiss (2009), p. 81.

<u>Authentic Mathematical Practice in the Work of Teachers</u>

To what extent do the mathematical activities commonly seen in classrooms reflect authentic mathematical work? Do current norms in mathematics instruction promote either mathematical proficiency or curiosity? Does the rationality that drives mathematics teaching help encourage an appreciation of mathematics?

Herbst and Chazan (2011) has suggested that it is crucial that we recognize how instruction typically works, understanding the practical rationality that underpins teaching, if we are to design reforms that are viable and sustainable. It is through incremental changes, which recognize current practice, that permanent transformation is most likely to occur, but how might incremental changes be introduced? What form might such changes take?

The key role of problem posing in mathematics instruction has long been recognized. Silver (1994) noted that problem posing is not only a prominent feature of mathematical activity; it also features heavily in "inquiry-oriented instruction" and can serve to create an environment in which students are more engaged.

Here we describe briefly how the five generative moves for problem posing (Fig. 1) could be relevant when describing the potential for secondary mathematics education to include instances of "authentic mathematical work". Suppose a high school geometry class has been studying the properties of triangles, and has found (either through empirical exploration, deductive proof, or a combination of the two) that the three angle bisectors of any acute triangle always intersect in a single point. The following scenarios show how instructional interventions can change the direction of the task and have the potential to depart from normative geometry instruction.[6]

- One possibility is that the teacher might ask, "Does it really matter whether the triangle is acute or not?" Investigating this question could lead the class to the conclusion that, in fact, the initial restriction to the case of acute triangles was unnecessary, and that the conclusion obtains

[6] The end goal is not for the instructor to make such interventions, but that all classroom participants, including students, begin to adopt this problem posing mindset.

for all triangles — a case of *weakening the hypothesis*, the first generative move in Fig. 1.

- Another possibility is that the teacher might encourage the class to seek to *strengthen the conclusion* of what has been proven, for example by providing additional properties that characterize the intersection point of the three angle bisectors of a triangle such as offering, "Not only do they intersect at a single point, but that point is the center of a circle that can be inscribed in the triangle."
- A third possibility is that the class might seek to *generalize* their findings, for example by asking, "What happens if you construct the angle bisectors of *other* polygons? Do they meet at a point, and if not, what do you get?"
- A fourth possibility is that the class might seek to *specialize* their findings, for example by observing, "If you do this with an *equilateral triangle*, there seems to be more than can be said about the resulting figure — for example the point of intersection seems to equidistant from the three *corners* of the triangle as well."
- A class that has observed this last property might then *consider the converse* question: "If the angle bisectors of a particular triangle meet at a point that is equidistant from the three corners of the triangle, does that mean that the triangle in question *must* be equilateral?"

The examples above illustrate how the generative moves identified in Weiss (2009) can be used to describe and promote the practice of *wondering mathematically* about what is true, a core component of authentic mathematical practice. More examples could be generated *ad lib* by iterating and recombining these moves. For example, the generalization to the case of other polygons could lead to a subsequent specialization to the case of quadrilaterals (which in turn could be subsequently refined to the case of various "special quadrilaterals"). The many variations on this "angle bisector problem" have played a key role in the representations of mathematics teaching used by Herbst and his collaborators as probes of geometry teachers' practical rationality (see Aaron, 2010; Herbst & Chazan, 2006; Weiss & Herbst, 2007; Weiss, 2009).

Authentic Mathematical Practice in Teacher Education

Many of the norms that characterize contemporary mathematics education are at a great distance from authentic mathematical practice. Herbst and Balacheff (2009) argue that an appeal to the disciplinary obligation can, in some cases, provide grounds for departing from those norms. This, however, requires that teachers hold a fuller and more nuanced view of authentic mathematical practice. In this section we address the role of teacher education in cultivating such a view.

Ball (1988b) identified a number of widespread views among preservice teachers, including "Mathematics is a mostly arbitrary collection of facts," "Doing mathematics means following set procedures," and "Doing mathematics means using remembered knowledge and working step-by-step" (pp. 104-108). Her findings showed that preservice teachers predominantly view mathematics as a "closed" field, one in which there are no new questions left to ask. When asked to respond to the statements "Some problems in mathematics have no answers" and "There are unsolved problems in mathematics", the preservice teachers in Ball's study expressed confusion. For them, "wondering mathematically" simply does not exist as an activity.

The impact of these views of mathematical practice is significant. In a recent study, Cross (2009) showed that teachers who understand mathematics to be primarily about "formulas, procedures, and calculations" consistently defaulted to an initiate-respond-evaluate pattern in their interactions with students. In contrast, teachers who regard mathematics primarily as being about the "thought processes and mental actions of the individual" were more likely to engage their students in extended, continuous discourse (Cross, pp. 332-3). Cross concludes that teachers who do not hold beliefs consonant with supporting "learner-oriented classroom environments" should be engaged in programs intended to transform their beliefs.

The responsibility for cultivating an awareness of authentic mathematical practice in preservice teachers rests, by necessity, with teacher education. Mathematics teacher educators "have the dual responsibility of preparing teachers, both *mathematically* and *pedagogically* (Liljedahl, Chernoff, and Zazkis, 2007, p. 239)." Although many colleges and universities preserve an institutional separation between mathematics content courses and mathematics methods courses, undergraduate mathematics courses should not be the only opportunities for future teachers to develop a sense of and appreciation for authentic mathematical work. Learning to wonder mathematically can, and should, be a goal of teacher education courses. Experiences with mathematical discovery have been shown to have a profound, transformative effect on future teachers' beliefs about the nature of mathematics and its teaching and learning (Liljedahl, 2005). Mathematics teacher education should make the processes and mechanisms of problem posing (including the generative moves of Table 1) explicit, and draw attention to how they can be used to navigate productively through open-ended problem spaces. Through engagement in, and explicit attention to, such mathematical activities, teachers might come to view mathematics differently. If they come to view mathematics differently, the disciplinary obligation that partly frames their instruction could lead to changes in what they deem valid representations of mathematics.

Besides implementing tasks that model authentic mathematical practice, mathematics education classes could provide future teachers with exposure to examples of the rich mathematical thinking that students are capable of and often bring to the classroom. Mathematics education classes should also help future teachers consider how to value and capitalize on students' wondering as well as how to promote problem posing by and mathematical curiosity in their students. Future teachers need exposure to and interaction with representations of classroom instruction (like case studies, videos, animations, etc.) that model authentic mathematical practice. Ideally teacher educators should be able to provide both actual and hypothetical episodes of instruction to show both what is currently possible and being done as well as foreshadowing what might be possible if current norms were questioned.

Mathematics educators could provide future (and also current) teachers opportunities to witness episodes of instruction that depart from normative practice but that exemplify authentic mathematical work. For teacher educators, a direct encounter with teachers' reactions to such breaches can help make visible the (usually tacit) norms that guide the rationality of teaching. These encounters have the potential to shape or transform teachers' views of the nature of mathematics and its teaching and learning.

Conclusions

The mathematics education community has a long history of efforts to improve teaching, and yet teaching remains largely resilient in the face of reform. One possible reason for this difficulty is that teacher education has struggled to instill a mathematical sensibility in preservice teachers, many of whom have little or no direct experience with authentic mathematical practice. A second possible reason for this difficulty is that reform efforts often fail to consider the norms that drive and sustain the practice of mathematics teaching as it exists currently. A strong case can be made for the use of practical rationality as a lens for viewing both research and teacher education: if we are to design reforms that are viable and sustainable, it is crucial to understand the practical rationality that underpins teaching (Herbst & Chazan 2011).

It may be somewhat naïve to expect that, simply by providing preservice teachers with opportunities to experience authentic mathematical practice, we will somehow transform them into a different kind of teacher, one who creates opportunities for his or her own students to engage in such practices. On the other hand, it seems to us unassailable that such preservice teacher education is a *necessary*, even if not sufficient, condition for such an outcome. It is almost impossible to imagine teachers engaging students in the processes of wondering mathematically, when the teachers themselves have never experienced such activity. Cultivating a richer vision of mathematics as a discipline may make it possible (although by no means certain) that teachers can, in the future, appeal to the disciplinary obligation as grounds for change.

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