Problem-Based Learning in Mathematics

By Thomas C. O'Brien (posthumously)
with Chris Wallach and Carla Mash-Duncan

A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations, he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some independent means of, independent thinking.

George Polya, preface to the first edition, *How To Solve It*, Princeton University Press, 1945.

For years problem-solving has been an aspect of the American school mathematics curriculum. But for most children— contacts with math educators around the country suggest 80 to 90 per cent of children—problem solving is limited to "word problems", i.e. computational exercises couched in words.

Word problems are a pretty narrow subset of the universe of problems. We can say with some authority that we have not solved a word problem outside a math classroom in many decades.

A more general definition of "problem" is a situation with a goal and the means to the goal is not known in advance. As the great mathematician George Polya said, [private conversation] "A problem is when you are hungry late at night and you go to the refrigerator and the refrigerator is empty. *Then* you have a problem."

In 2000 the National Council of Teachers of Mathematics' Principles and Standards for School Mathematics defined problem solving as follows:

> *Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on*

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> *their knowledge, and through this process, they will often develop new mathematical understandings*. [Note 1]

Polya suggested two aims for elementary school mathematics. First are the "good and narrow aim of education."

> *… the good and narrow aim of the primary school: to teach the arithmetical skills — addition, subtraction, multiplication, division, perhaps a little more. Also to teach fractions, percentages, rates, and perhaps even a little more. … Arithmetical skills, some idea about fractions and percentages, some idea about lengths, areas, volumes, everybody must know this. This is a good and narrow aim of the primary schools, to transmit this knowledge, and we shouldn't forget it.*

And then there is a higher aim:

> *But I think there is one point which is even more important. Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. To solve certain problems of multiplication or addition, this belongs to the good and narrow aim. To the higher aims about which I am talking now, is some general tactics of problems. To have the right attitude to problems. To be able to be prepared to attack all kinds of problems — not only the very simple problems, which you can right away solve with the skills of the primary school, but more complicated problems, problems of engineering, physics and so on. This will be, of course, farther developed in the high school, still farther for those who take a technical profession at the university, but the foundations should be prepared already in the primary school. And so I think an essential point in the primary school is introduce the children into the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude, a general aptitude to the solution of problems. Well, so much about the general aim of the teaching of mathematics on the primary school level. [Note 2]*

Since Polya's death in 1985 there has been a burgeoning movement involving problem solving as a fundamental aspect of education which incorporates and goes beyond the development of problem solving tactics and attitudes.

Problem solving has become to be seen as a method of *causing learning to take place.*

At the heart of problem-based learning (PBL) is collaborative work among students in devising and solving problems involving conceptually complex material. [Note 3]

PBL, said to have been originally developed for the training of physicians at McMaster University in the late 1960s, has been incorporated into over sixty medical schools and other health-related programs such as nursing, dental and veterinary schools. Moreover, PBL is said to have been adopted by numerous disciplines including business, chemistry, biology, physics, mathematics, education, architecture, law, engineering, social work, history, English and literature, history, and political science. [Note 4]

The implementation of problem-based learning (PBL) entails not only the re-design of curriculum but also the development of effective facilitation-cum-coaching approaches. PBL curricula innovation typically involves a shift in three loci of educational preoccupation: from what content to cover to what real-world problems to present; from the role of lecturers to that of coaches; and, from the role of students as passive learners to that of active problem-solvers and self-directed learners. [Note 5]

What does this have to do with the mathematical education of children? PBL, it seems to us, is intimately related to the Piagetian notion that knowledge is a personal construction, not a set of fixed entities transmitted to be stored until text time. In classrooms, this means that interesting tasks, problems, and investigations should be actively engaged by learners.

The British mathematician Alfred North Whitehead hinted at PBL when he said, 90 years ago, "In training a child to activity of thought, above all things we must beware of what I will call 'inert ideas'—that is to say, ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations." [Note 6]

It is a complex task but teachers need to find out where the learner is in order to challenge the learner with problems and investigations which have a moderate mismatch with the learner's present status. Thus challenged, the child will revise or extend or generalize his/her present fabric of ideas and relationships. This is what learning is all about, not the storage, rehearsal, and production-upon-command of inert facts.

Our task as educators is to come up with appropriate provocation, i.e., good problems and investigations to engage children's minds and imaginations.

That is to say, much of learning takes place by *provoked adaptation*. This is a message especially appropriate to mathematics education.

Recent Work with Children

During the past five years or so O'Brien has worked in with local teachers in elementary school math classrooms. The work was undertaken from a provoked adaptation point of view (which we now know is intimately related to PBL). That is, no teaching took place, problems were posted for children working collaboratively, and children were almost universally successful in their work.

Not the least, children's enthusiasm was such that we sometimes had to exert "crowd control" in the sense of giving children poker chips (two to each child) to be spent to in order to address the entire class, so energetic was their desire to share their findings.

In general, the tasks involved necessary inference—an utterly basic aspect of mathematical thinking—and in general the problems involved games devised by the author. By "inference" is meant the deriving of new information from old information.

(Suppose I hide a penny in one fist and don't tell you which fist, I show both fists, closed, to you. You choose one of the fists and find out that it is empty. You can't see it, but your mind can see that the penny is on the other fist.

(Or suppose I show you 8 pennies. You count them. Then I ask you to close your eyes and I cover some of the pennies with my hand. I ask you to open your eyes and tell me how many pennies are under my hand, You relate the three classes of chips—the original chips, the showing chips and the hidden chips—to *infer* the number of pennies under my hand, Interestingly, many teachers will predict that 5 and 6 year olds will subtract to get an answer. They don't.) [Note 7]

The results have been widely reported in the US and the UK. [Note 8]

The latest work was undertaken with first and fourth grade children in a private school in the midwestern USA.

First Grade

The activity is called Mystery Person. It was invented, so far as we know, by O'Brien.

A number of people are asked to sit in a circle and their initials are drawn on a large paper pad that everybody can see. In the diagram that follows, C is for Charles, etc.

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    C
B       N
R       L
    T
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The teacher secretly writes down the name of a Mystery Person. The players have to gather clues and infer who the Mystery Person is.

They ask the person who chose the Mystery Person about a particular person. If the person is the Mystery Person OR if it is next to the Mystery Person, the feedback from the teacher is "Hot." Otherwise, the feedback is "Cold."

The reader is asked to play this game with one or more adults. Then turn the tables; a different player hides the Mystery Person and the person who originally hid the Mystery Person has to gather clues and find the Mystery Person.

Once the reader has played the game several times, the challenge is to solve these problems with the list of people C-N-L-T-R-B as configured above. Mathematics is not a spectator sport.

In the circle above, C is cold. T is Hot.

Is there enough information to figure out who the Mystery Person is? If so, who is it? If not, what question would you ask next? The answers are given at the end of this article.

Up to now, you have played Mystery Person with one person hidden. We ask readers to go back to the C-N-L-T-R-B configuration above and challenge friends to a game.

We tried the Mystery Person game with first-graders at the beginning of the school year and were pleased to see that they succeeded. They enjoyed the games so much that Tom was accosted by a stranger in a super market. "You're working with my little Jamie with the Mystery Person game, yes? I want you to know that Jamie loves the games and insists that we play the games at home around the dinner table."

We stayed with the one-person game for two sessions, each about 25 minutes, and it was clear that the children were successful. There was rarely a wasted question and children knew when a conclusion had been reached. Children worked together enthusiastically and cooperatively. It was also clear to Chris that at their young age and at this early time in the school year, their attention span was such that that they needed a change of pace and so we took a break for other activities.

It was not until January that we got back to the Mystery Person games. We had done similar work with fifth graders in the past (Cite "What is Fifth Grade?) and we were keen to find out how children at this age would do with two Mystery Persons.

The group, as before, was Chris's math class, 14 children selected from three first grade classes in the school.

Chris asked 7 of the children to sit in a circle on the floor and she asked the rest of the children to sit in chairs in a circle surrounding them. She put the inner-circle children's names on the board.

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    J
S       K
Ke     L
  E   C
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First we played for one Mystery Person. Tom secretly wrote the name of one of the children in the inner circle and gave out data while Chris selected children from the entire group to ask questions and explain why they were asking about this or that person.

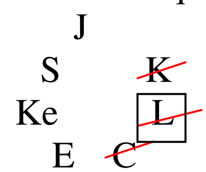
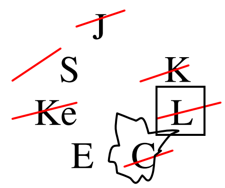
The only bit of "teaching" that took place, aside from reminding children of the rules of the game at the outset, was to ask children to note the consequences of the data they were given.

But Chris had an extra arrow for her bow. She commonly asked children to explain their thinking to her and to the class. And, unlike many American classrooms where the teacher moves on once a correct answer or a sensible explanation is given by one child, Chris asked several people to share their solutions and/or their thinking and often she chose a child whose solution was weak or incorrect. "What do you think of that?" was Chris's question to the class. Never did Chris say or imply that a child was incorrect.

Here is the way the game went. The consequences were placed in the pad by children taking turns.

A ring of fire meant hot. An ice cube meant cold.

A check meant that the person was a possible Mystery Person and an X meant that they had been ruled out as a Mystery Person.

Child	Chris	Consequences
1. Tell me about L?	L is cold	
2. C?	2. Hot	

3. We're finished, It has to be E.

This was for warm-up. (Noteworthy, only this one game was needed.) We were pleased that, unlike much of the school curriculum, children were successful four months later.

Chris exchanged inner- and outer-circle children and the game went like this: (We won't provide a Consequences chart in order to provoke readers into following children's tactics.)

H C
I Je
R L
N

Child	Chris
1. Tell me about L?	L is hot.
2. C?	Hot
3. Je?	Hot
4. R?`	Cold
5. H?	Hot
6. I?	Hot

7. We only need one more question and the game is finished,

What do you think about children's thinking? Did you infer both Mystery Persons? Try some of this with kids?

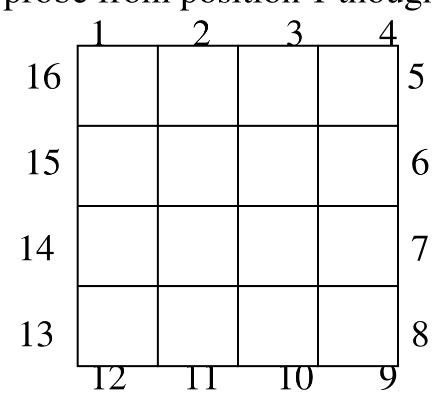
Fourth Grade

We worked with Carla's class of 14 fourth graders for 50-minute sessions for ten or so Thursdays staggered throughout the year by Tom's travel and school holidays and events.

For several of these sessions we worked with a game called Find It, also invented by Tom. [Find It is available for Palm PDAs from Handango: See http://www.professortobbs.com/software.htm]

As with the first graders, the sessions involved the whole group, with children encouraged to work out certain issues (such as "What's the best place to start? What are the consequences? What's a good next step?) in small groups.

Find It involves a 4 by 4 grid. Players can opt for 1-12 diversions to be placed randomly in the grid and the task is to infer where the diversions are. The player launches a probe from position 1 though 16 to look for the diversion.



There are three games, Righties, Righties and Lefties, and Randoms. In the Rightie game, a probe makes a right turn when it hits a diversion. Righties and Lefties are a random mix of the two types of diversion and Randoms are randomly Righties or Lefties.

When a probe is launched, the destination and the number of diversions are reported. For example, suppose a player is playing Righties and has chosen that only one Rightie be hidden.

And suppose that the player shoots Probe 1 and finds that it exits at 12 with 0 hits.

This means that no Righties have been encountered. And thus four boxes in the grid can be eliminated.

	1	2	3	4	
16	x				5
15	x				6
14	x				7
13	x				8
	12	11	10	9	

Suppose the next shot is 16. And suppose the probe exits at 11 with 1 hit. You know with logical necessity that there is a Rightie in the 16-2 (or 16-11) box, The game is finished.

Here is a game involving 5 Righties. Can you locate the Righties? The answer is given below. This game took fourth graders 13 minutes to solve.

Start	9	10	11	12	16	1
Exit	7	3	6	9	1	15
Number of Hits	1	0	1	2	3	1

After two or three 20-minute sessions with Righties, the children were both efficient and confident. They had equilibrated. As is the case in most situations involving equilibration, they wanted to move higher.

Here are the data for 12 Righties. Fourth graders took 20 minutes. The answer is given below.

Start	1	2	3	4	6	7	8	11	10	12	14
Exit	16	1	2	5	4	6	9	10	8	11	14
Number of Hits	1	2	2	3	2	2	3	2	1	2	6

Summary

The results we report here are consistent with the previous five years' work. Children constructed important ideas in the face of a problem situation. They did so collaboratively, they respected one another's thinking, and their overall enthusiasm and eagerness to go further was at all times impressive.

The results here are consistent with the constructivist notion that moderate conflict (i.e., a problem which involves a moderate mismatch between the learner's original network of ideas and abilities and those needed to solve the problem) leading to provoked adaptation is at the heart of learning.

Perhaps most important, the activities here go somewhere. Polya said [private conversation], "First, a good problem must be difficult enough for the student, else it is an exercise and not a problem. Second, it should be interesting to the student. And most, important, it should *go somewhere.* Inference is an important "somewhere. " It is the glue that holds mathematics—and in fact, much of life—together.

The results here are consistent with the principles of problem-based learning.

Certainly problem-based learning is not entirely new to math teachers. Surely some teachers have used the principles of PBL in their classrooms from time to time, but no concerted and continuous thrust has been given PBL in American mathematics education in either research or practice.

The fact that PBL has been used widely and apparently successfully in a wide variety of fields is heartening. It is reasonable to suspect that leaders in a wide variety of disciplines, including medicine, do not adopt new polities and practices without good reason.

More important, the results are consistent with the Piagetian emphasis on equilibrium. Equilibrium and homeostasis are fundamental not only to the biological world but to the world of education.

Perhaps this is the time for American mathematics education to make some small starts away from the parrot-training that is so common and so fruitless.

Notes

1. National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics, p.51, Reston, VA: NCTM

2. http://www.mathematicallysane.com/analysis/polya.asp

3. Karl A. Smith, The Role of Collaboration in Designing and Practicing Problem Based Learning,
http://www.udel.edu/ce/pbl2002/speaker_smith.html

4. http://www.uc.edu/pbl/intro_history.shtml

5. Oon-Seng Tan, "Key Cognitive Processes in PBL Practices: Insights for PBL Facilitators."
http://www.udel.edu/ce/pbl2002/speaker_tan.html

6. The essay was first published in 1917. See Alfred North Whitehead, The Aims of Education and Other Essays, (New York: Free Press, 1957).

7. O'Brien, Thomas C., and Richard, June V. "Interviews to Assess Number Knowledge," The Arithmetic Teacher, May 1971.

8. Thomas C. O'Brien and Judy Barnett, "Fasten your seat belts," Phi Delta Kappan, 85(3), 201-6, November 2003.

Thomas C. O'Brien and Judy Barnett, "Hold on to your hat," Mathematics Teaching, 87, June 2004.

Thomas C. O'Brien and Chris Wallach, "Children Teach a Chicken," Mathematics Teaching, 93, December 2005.

Thomas C. O'Brien, "A Lesson on Logical Necessity," Teaching Children Mathematics, 13(1), August 2006.

Thomas C. O'Brien and Chris Wallach, "Children's Construction of Logical Necessity," Primary Mathematics, Autumn 2007.

Answers

1. In the one-person Mystery Person game, the data are inconclusive.

2. Children's work (including answers) on the 5 Righties and 12 Righties tasks is shown below.