

## **Rules Without Reason: Allowing Students to Rethink Previous Conceptions**

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*Abstract: This paper reports on the strategies chosen by a group of sixth-grade students in an urban informal learning program as they worked to solve an open-ended, non-routine task. In particular, the paper focuses on the ability of these students to rise above their previous, procedure-based misconceptions and arrive at a mathematically reasonable solution. Aspects of the problem task, the problem-solving environment, and, importantly, of the nature of the teacher's interventions are analyzed to determine the conditions that encouraged students to approach mathematics as a logical, meaningful, sense-making activity.*

**Keywords:** Cuisenaire™ rods ; conceptual versus procedural; fractions; meta-cognition; open-ended tasks; problem solving; procedural learning; sense making; student reflections; teacher interventions

### **Introduction**

The goal of building students' mathematical reasoning and their ability to create and defend proofs is a priority among mathematicians, mathematics educators, and policy leaders. Researchers concur that reasoning and proof are the foundation of elementary

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and middle-grades mathematics learning and are necessary for building sustainable mathematical understanding (Hanna, 2000; Hanna & Jahnke, 1996; Polya, 1981; Stylianides, 2007). Ball and Bass (2003) contend that “the notion of mathematical understanding is meaningless without a serious emphasis on reasoning” (p. 28). They argue that understanding is founded on reasoning in that students must use reasoning to understand relationships and make connections to new ideas.

Unfortunately, students struggle with reasoning, especially reasoning involving analysis of relationships between quantities, rather than just reasoning of numbers or operations alone (Kouba & Wearne, 2000). They also experience difficulty when explaining and justifying their thinking (Arbaugh, Brown, Lynch, and McGraw, 2004). Although students may be able to solve complex problems, they are not always able to cognitively defend their solutions, explain, or justify the process they used to reach an answer. This struggle is compounded by the fact that teachers often focus primarily on addressing the learning of “mathematics facts and concepts” and on “learning skills and procedures needed to solve routine problems” (Silver, Alacaci, & Stylianou, 2000, p. 339).

In addition, students’ knowledge that is not fully developed or not understood often hinders their ability to reason. When ideas that are not fully understood or developed are used in arguments, they can become obstacles that encumber correct reasoning. In the process of making sense of a problem situation or creating an argument to support a solution, conflicts between incomplete knowledge and new knowledge schemes emerge. In these cases, students need to be open to adapt their newly developed knowledge to accommodate their new understandings. This can come about in situations where students can talk about and explain their reasoning as well as hear the explanations of other students (Wearne & Kouba, 2000, p.189). Encouraging communication in problem solving can promote students’ growth in mathematical understanding and in solving more complex problems. With support from others, students can extend their knowledge, sometimes by building alternative representations, and by sharing, discussing and arguing about their ideas.

In this paper, we share an episode from an after-school mathematics program where a group of sixth-grade students were prompted to rethink what they know about fraction “rules”. They did this by building their own evidence and convincing themselves and others to believe in their power to reason.

## **Theoretical Framework**

### Understanding

Robert B. Davis (1992) explains that, given opportunities, students will create their own ways of understanding and build representations and understanding based on their previous knowledge and experiences. However, Davis points out that what students learn is built upon this foundation of understanding and that future learning may also be limited by previous understanding. Davis suggests that understanding is achieved when one is able to fit a new idea into a larger structure of earlier constructed ideas. Davis’ (1992) theory of understanding notes, “One gets a feeling of understanding when a new idea can be fitted into a larger framework of previously-assembled ideas” (p. 228). Davis refers to the representational structures that a learner builds as a collection of *assimilation*

*paradigms*. Davis and Maher (1993) describe these assimilation paradigms as the act of fitting new ideas into larger frameworks of previously assembled ideas. The learner views the new idea as “just like” or “similar to” an existing experience and uses this to accommodate the new knowledge (Davis & Maher, 1993). Davis (1992) notes the existence of obstacles to understanding, which he refers to as “cognitive obstacles” and describes as “improperly chosen assimilation paradigms that lead to incorrect ways of thinking or that are limited in their scope” (p. 226).

Often, the mathematical instruction in schools does not connect to children’s natural, experience-based understandings. Instead, it requires students to adapt their reasoning styles to fit those valued by schools (Malloy, 1999), which may result in the aggregation of cognitive obstacles. Too often, the traditional approach to teaching mathematical concepts emphasizes students’ memorization of rules and procedures and manipulation of symbols. Many of these rules are meaningless to children, having been learned by rote methods (Davis 1994). Erlwanger (1973) reports the case of Benny, a twelve-year-old boy in the sixth grade using Individually Prescribed Instruction (IPI), who believed that mathematics consists of different rules for different problems that were invented at one time but work like magic. In Benny’s eyes, mathematics was not a rational and logical subject where one has to reason, analyze, seek relationships, make generalizations, and verify answers; rather, it was a game where one discovers the rules and uses them to solve problems (Erlwanger, 1973). Benny created his own “rules” for adding fractions based on what he perceived as random procedures. Kamii and Diminck (1998) argue that teaching rules and conventions can be harmful because they cause children to relinquish their own ideas and disconnect the content from the concepts. When exposed to this kind of instruction, students often remember erroneous rules and procedures, as was seen with Benny. Kamii and Warrington (1999) propose that the focus of instruction should shift from teaching that emphasizes physical and social knowledge to that which values and encourages children’s own reasoning.

Yackel and Hanna (2003) concur and argue the view of mathematics as reasoning can be contrasted with the view of mathematics as a rule-oriented activity. Other researchers support the fact that the sole teaching of algorithms can be detrimental and counterproductive to the development of children’s numerical reasoning. Mack (1990) came to this conclusion after finding that algorithms often keep students from even trying to use their own reasoning. Through her work with eight sixth-grade students, she found that while all eight students began with serious misconceptions about fractions, they also had a considerable repertoire of informal knowledge that allowed them to solve real-life fraction problems. Mack found that students built on informal knowledge when given the opportunity to connect problems represented symbolically to real-life situations that they understood. However, faulty, isolated knowledge of rote procedures often got in the way of problem solving, since students often remembered erroneous algorithms and had more faith in these rules than in their own thinking. Overall, Mack (1990) concluded that students could use informal knowledge about fractions to build meaning and understanding. She noted that the knowledge of rote procedures can interfere with the building of fractional understanding and that the results give evidence to the argument that students should be taught concepts before procedures.

According to Skemp (1971), “. . .to understand something is to assimilate it into an appropriate schema” (p. 45). As a result, a student’s level of understanding depends upon

the schema he or she has created during instruction. Understanding develops as students form connections between new and old knowledge and create appropriate schemas to make sense of new knowledge (Davis & Maher, 1997). These schemas are built on previous understanding as students make connections between schemas. Often, students run into roadblocks that they must overcome by building alternative representations and sharing ideas. Students exhibit logical understanding when they are afforded the opportunity to justify their reasoning in a community of learners and are able to adapt previous [mis]understandings/beliefs. The learning environment is a crucial component of the successful development of reasoning, and in turn, it is necessary for the development of understanding. According to Martino and Maher (1999), if they are given a supportive environment, students will appreciate the value in discussing their ideas with classmates in order to help formulate and refine their own thinking.

### Communities of Learners

Vygotsky (1978) hypothesized that students internalize the discussions that occur in group contexts. Cobb, Wood, and Yackel (1992) built on this claim by stressing the role of social interaction in the construction of mathematical knowledge and calling attention to the role of discussions in a mathematical community. Cobb (2000) stresses the importance of studying students' mathematical activity in the social environment of the classroom. He states that students' mathematical understanding is based on the relationship between individual student activity and overall classroom practice. The classroom micro-culture has a significant influence on the meaning students make of mathematics, as well as their use of explanations.

According to Maher and Davis (1995), a learner is influenced by the representations created by others in his or her community. They explain that "[d]ifferent forms of reasoning coexist within this community and different forms of representation are used to transmit ideas, explore and extend these ideas, and then act upon them" (Maher & Davis, 1995, p. 88). Maher and Davis use a ribbon and bow metaphor to describe this community of learners, explaining, "A single ribbon may be viewed as a particular path pursued by a particular child in developing an idea; the bows, varied in complexity and size, might be viewed as ideas received from other members of the community and acted upon by individual children in such a way as to influence their development" (p. 88). A strong community of learners where ideas are transferred from one student to another exemplifies the bow metaphor.

According to Maher and Martino (1996, 2000), learners bring personal experience to every mathematical investigation. Given a context that includes a mixture of personal exploration and social interaction, coupled with students' mental representations, knowledge and beliefs may be refined and modified to build new ideas and theories. The responsibility shifts from the teacher to the learner, and therefore the learner has the opportunity to make sense of the mathematics through careful reasoning and the building of arguments (Maher, 2005).

The classroom micro-culture or community in which learning occurs shapes the meaning that students construct. A community of learners is created through mathematical discussions in a rich social environment. Within this community, ideas are shared, modified and agreed upon and students' understanding is built. Collaboration is often viewed as the support provided by learners within a community in the form of

missing pieces of information needed to solve the problem (Francisco & Maher, 2005). Francisco and Maher contend that a form of collaborative work that involves group members relying on each other to generate, challenge, refine, and pursue new ideas is also important (p. 369). With this type of collaboration, rather than piecing together their individual knowledge, the students work together to build new ideas and ways of thinking.

### **Methodology and Data Sources**

#### **Participants**

This research is a component of a larger ongoing longitudinal study, Informal Mathematics Learning Project, (IML)<sup>2</sup> that was conducted as part of an after-school partnership between a state university and a school district located in an economically depressed, urban area. The district's student population consists of 98 percent African American and Latino students. Our study focuses on the development of reasoning of middle-school students. In the IML program the participants met twice a week over a six week period in four cycles that each focused on different task strands (combinatorics, probability, algebra, or fractions). During each session, students were invited to work collaboratively on open-ended mathematical tasks. Task choice played an important role in the project's objectives, since if the tasks were too simple, the students' schemas would not be enhanced, but if they were too difficult, the students would be discouraged from engaging in an attempt to find solutions. We report on the first cohort of students, 24 sixth-graders, who, over five, 60-75 minute sessions worked on fraction tasks, interacted with peers, and had ample time to explore, discuss and explain their ideas. The sessions were facilitated by two of the researchers (university professors) who designed the program. For the purpose of this paper these two researchers will be referred to as teachers. The students worked in heterogeneous groups of four and participated in whole class discussions. In each session, problems were posed and students were asked to explore solutions in their small groups first and then to share with the whole class. They were invited to collaborate and discuss their ideas with one another, were encouraged to justify and make sense of their solutions, and were challenged to convince one another of the validity of their reasoning.

#### **The Tasks and Tools**

The strand of tasks was developed from an earlier research intervention with fourth-grade students. The fourth grade students had not been previously introduced to fractions. Through their engagement in the fraction tasks, it was documented that the students used reasoning to compare fractions, find equivalent fractions and perform operations on fractions after working on the tasks (Steencken & Maher, 2003). The current study differed in that the sixth grade students had already been introduced to fraction rules and procedures with little conceptual understanding of meaning. We therefore chose to give the sixth-grade students tasks that focused on rational numbers and the use of manipulative materials to build concrete models.

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The students were given Cuisenaire™ rods for building models to represent their solutions. A set of Cuisenaire rods, as shown in Figure 1, contains 10 colored wooden or plastic rods that increase in length by increments of one centimeter.

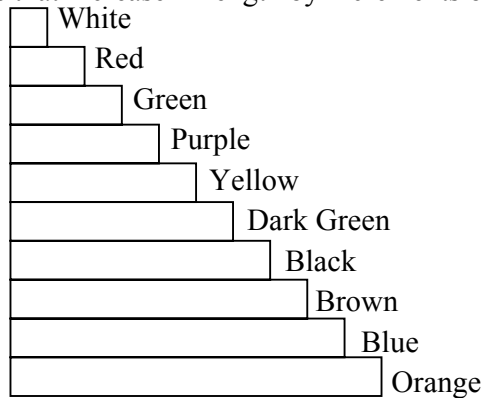


Figure 1. “Staircase” Model of Cuisenaire Rods.

As part of introducing students to working with the rods, the researchers explained that the rods are given permanent color names. These names, along with their respective lengths, are: white (1 cm); red (2cm); light green (3 cm); purple (4 cm); yellow (5 cm); dark green (6 cm); black (7 cm); brown (8 cm); blue (9 cm); and orange (10 cm).

The rods were assigned variable number names and students were challenged to identify fractional relationships with respect to a specified given unit. For example, one task was: “*What number name would you give to the dark green rod if the light green rod is called one? Discuss the answer with your group*” (Maher, 2002). Students were then provided time to investigate their solutions and make claims in their small groups. They were encouraged to use the rods to build models as justifications in support of their solutions. After being afforded ample time to build models, collaborate, and justify their solutions, groups were invited to the overhead projector to share their findings with the class. During these whole group discussions, students shared their findings, challenged each other, and often reflected on and revised their solutions.

#### Data Collection and Analysis

Each session was videotaped with four different camera views. The cameras focused on different groups of students and one of the cameras also captured the presentations at the overhead projector. Video recordings and transcripts were analyzed using the analytical model outlined by Powell, Francisco & Maher (2003). The video data were described at frequent intervals; critical events (episodes of reasoning) were identified and transcribed, and codes were developed to flag for solutions offered by students and the justifications given to support these solutions. Arguments and justifications were coded according to the form of reasoning being used, direct or indirect, and as valid or invalid, based on whether or not the argument started with appropriate premises and the deductions within the argument were a valid consequence of previous assertions. Students’ construction of solutions and their subsequent justifications were then traced across the data in an effort to document and analyze their journey to mathematical understanding.

## **Results**

Throughout the intervention, the teachers were careful to refrain from correcting students' errors, rather allowing the students to discuss their ideas until their classmates either refuted or agreed with their solutions and justifications. Upon analysis of the data, an interesting pattern was noticed. As the sessions progressed, students were observed to confidently share their solutions, both correct and incorrect, and other students often challenged these solutions and countered with refutations of the justifications. In particular, episodes during which a student provided an invalid solution were likely to evidence rich student interaction, varied forms of reasoning, and the emergence of a soundly-justified, correct solution.

An episode from the fourth session of the after-school program is used to illustrate the ways in which students overcame previous misunderstandings while attending to open-ended tasks. In the previous session, the blue rod was given the number name one and students were asked to find a number name for the white rod and the red rod. During session four, students initiated naming the remaining rods in the set (when the blue rod was named one). Students first worked in small groups and then shared results with the larger community. An excerpt from each forum is provided below.

### **A Group of Three: Chanel, Dante and Michael**

Chanel lined up five white rods next to the yellow rod and used direct reasoning to name the yellow rod five-ninths. She then initiated the task of naming all of the rods using the staircase model (see Figure 1). She named the remainder of the rods, using direct reasoning based on the incremental increase of one white rod or one-ninth and used the staircase model as a guide and named the rods until she arrived at the orange rod. As she was working she said the names of all of the rods, "One-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths, ten...—wow, oh, I gotta think about that one, nine-tenths". Chanel showed Dante her strategy of using the staircase to name the rods and explained the dilemma of naming the orange rod to Dante, "See this is one-ninth, two-ninths, three-ninths, four-ninths, five-ninths, six-ninths, seven-ninths, eight-ninths, nine-ninths - what's this one [the orange rod]?" Dante replied, "That would be ten-ninths. Actually that should be one. That would start the new one". He initially named the orange rod ten-ninths but corrected himself and said that the orange rod would "start a new cycle"; and named it one-tenth. Michael named the orange rod a whole and explained that it was equivalent to ten white rods and Chanel agreed.

Chanel: It should be called a whole.

Dante: This is one, this is nine-ninths also known as one. This should be blue and this would start the new one – would be one-tenth.

After students worked for about five minutes drawing rod models, Dante told the group that that he heard another group calling the orange rod ten-ninths.

Dante: Why are they calling it ten-ninths and [it] ends at ninths?  
Michael: Not the orange one. The orange one's a whole.  
Dante: But I'm hearing from the other group from over here, they calling it ten-ninths.  
Michael: Don't listen to them! The orange one is a whole because it takes ten of these to make one.  
Dante: I'm hearing it because they speaking out loud. They're calling it ten-ninths  
Michael: They might be wrong! ...  
Chanel: Let me tell you something, how can they call it ten-ninths if the denominator is smaller than the numerator?  
Dante: Yeah, how is the numerator bigger than the denominator? It ends at the denominator and starts a new one. See you making me lose my brain.

As the students were working a teacher joined their group. Dante shared his conjecture, "It's the end of it and it starts the new one to one-tenth because the blue ends it and so the orange starts a new one just like - pretend there were smaller ones than just a white. So this would be considered like blue, a one". The teacher reminded him that the white rod was named one-ninth and that this fact could not change. Again she asked him for the name of the orange rod and he stated, "It would probably be ten-ninths". When prompted, Dante explained that the length of ten white rods was equivalent to the length of an orange rod. The teacher asked Dante to persuade his partners.

Chanel: No, because I don't believe you because –  
Michael: I thought it was a whole.  
Dante: But how can the numerator be bigger than the denominator?  
R1: It can. It is. This is an example of where the numerator is bigger than the denominator.  
Chanel: But the numerator can't be bigger than the denominator, I thought.  
Michael: That's the law of facts.  
R1: Who told you that?  
Chanel: My teacher.  
Dante: One of our teachers  
Michael: That's the law of math.

In the above dialogue, we see that even though Dante named the orange rod ten-ninths, using previous knowledge (of the name of the white rod) and a concrete model, he still questioned his answer. His prior understanding of the "rule" was so strong that he questioned himself even after building a concrete model and explaining the concept.

### The Whole Class

At the end of the session groups were asked to share their results with the class.. Malika and Lorrin named the orange rod ten-ninths and reported that they initially thought the numerator could not be larger than the denominator.

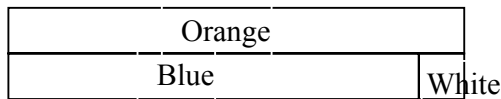
Lorrin: Because, before, we thought that because we knew that the numerator would be larger than the denominator and we thought that the denominator always had to be larger but we found out that that was not true. Because two yellow rods equal five-ninths, and five-ninths plus five-ninths equal ten-ninths.

Kia-Lyn and Kori explained that the orange rod had two number names.

Kia-Lynn: We found that ... the orange one has two number names. So because the orange one and the blue one – I thought that – our group had found out - that the orange is bigger than the blue one but when you add a one-ninth, a white rod, to the blue top it kind of matches. It kind of matches and we found out that you can also call the orange rod one and one-ninth.

Kori: So we were saying that if this [blue] is called one –

Kia-Lynn: It's also called one – um ten-ninths as Malika and Lorrin had said. But if you have one...white rod and you add it to the blue, it's one-ninth plus one is one and one-ninth and so if the blue rod and one white [they are using overhead rods to show a train of the blue rod and a white rod lined up next to an orange rod as shown in figure 2]. If you put them together then this means that it's ten-ninths also known as one and one-ninth



*Figure 2.* Kia-Lynn's model for naming the orange rod one and one-ninth.

Finally, Dante presented his strategy:

Dante: Well all I did was start from the beginning – start from the white – and you and all the way to the orange and what – like Kia-Lynn's group just said - I had found a different way to do it. Because all I - I had used an orange, two purples, and a red and since these two are purple and this is supposed to be purple but I had purple and I used a red since four and four are eight so which will make it eight-ninths right here and then plus two to make it ten-ninths. [He builds the model shown in figure 3 on the OH] That's what I made.

R2: So it's another way of showing that orange is equivalent to ten-ninths?

Dante: Um hum. And then I just did it in order – then the one I did right here – I just did it in order of whites by doing ten whites.

Orange		
Purple	Purple	Red

*Figure 3.* Dante's model for naming the orange rod ten-ninths

### **Discussion and Implications**

Specific factors in the after-school session enabled the students to challenge and revise their ways of thinking about mathematics. These factors included, but were not limited to, the following: challenging, open-ended tasks that invite students to extend their learning as they create and justify solutions, the opportunity to build models using concrete materials, the promotion of student collaboration in small groups and the opportunity to share ideas in the whole class forum, strategic teacher questioning, and the portrayal of student as determinant of what makes sense.

The tasks were open-ended such that students could expand on a given task, as in the above episode when the students initiated naming all of the rods in the set. In addition, the nature of the tasks and the time allotted for exploration allowed students to work at their own pace and readiness level. As the other students grappled with convincing themselves that ten-ninths was a viable number name, Kia-Lyn and Kori took the task to the next level and showed that ten-ninths is equivalent to one and one-ninth. In an environment that allowed students to act as teachers and present new concepts to their peers, the students were exposed to new ideas in a manner that was conducive to their assimilation.

In addition, the students were provided tools to build models and therefore they could conceptualize the fraction relationships. The Cuisenaire rods offered a concrete, visual model of ten-ninths and the students were thereby provided the means to show, using concrete evidence that this fraction did indeed exist. At the end of the session, Kori and Kia-Lynn presented a concrete model to show that these two fractions were equivalent and further justified this solution. These experiences offered the class new schema upon which to build and make connections (Davis & Maher, 1997) and the students were empowered to use these ideas and apply them to similar mathematical problems. Students, afforded the opportunity to explore, become convinced of, and justify their revised ways of thinking, became owners of the mathematical ideas they created. This ownership gave them the confidence to build new ideas and relationships.

The physical environment promoted student collaboration and it was further encouraged by the researchers asking students to listen to each other's ideas and to judge the merit of each other's justifications. Importantly, the researcher's careful questioning prompted students to explain their reasoning and invite their classmates to evaluate their thinking. When Chanel first grappled with naming the orange rod the teacher suggested she share her dilemma with Dante. After being afforded more time to think about the

task, Dante was asked to explain his thinking. Rather than correcting Dante, the teacher reminded him of the facts that were already established (that the white rod was named one-ninth). This subtle prompt enabled him to revise his thinking through the use of his own reasoning. Dante was then asked by the teacher to explain his thinking and convince his partners that his reasoning was correct. By working to convince his partners, Dante was able to reaffirm his reasoning and further convince himself of its validity.

Dante was again encouraged to have confidence in his own thinking during the second phase of the activity. After students were provided the opportunity to explain their thinking and discuss their ideas in their small groups, they participated in a whole class discussion, providing the opportunity for them to validate their ways of reasoning about the problem. Further, the arguments presented by others introduced them to alternative models and justifications. Although Dante used the staircase model to incrementally increase the names of the rods by one-ninth as he worked with his partners, he chose a different representation during his whole class presentation. After viewing the other presenters and listening to their presentations, his thinking was validated and thus he expressed confidence in his solution. This confidence led him to show two alternative models for naming the orange rod.

Malika and Lorrin shared that they, too, had previously believed that the numerator of a fraction could not be larger than the denominator; however, their reasoning and the concrete evidence that they used to show that five-ninths plus five-ninths is equivalent to ten-ninths was a stronger influence on their ultimate decision. In an environment that encourages reasoning, these students learned to trust their own logical ability and were thereby able to challenge and rethink their earlier understanding.

Overall, the teacher encouraged the students to revise their thinking and overcome their misconceptions. This was done via open-ended tasks, concrete materials, physical environment, and strategic questioning. More importantly, we can learn from what the teacher did *not* do during this session. The teacher did not correct the students and tell them that the rule that they had erroneously recalled was incorrect. Instead, she presented them with facts and allowed Dante to convince his group of the truth of his argument. Instead of replacing one teacher-initiated rule with another, the students were invited to resolve their differences and together agree on the mathematical validity of their solutions.

The teacher allowed the students to replace the previously learned rule by giving them the opportunity to independently build their own schemas for improper fractions. Instead of telling the children that ten-ninths is equivalent to one and one-ninth and showing them the proper algorithm to make this conversion, the teacher allowed the students to discover this on their own, in a setting that encouraged discussion and argumentation, and one that guaranteed students' safety as they offered their ideas for critique by their peers.. When students are encouraged to reason and thereby become members of a community of engaged, active learners, they are able to exhibit understanding and build trust in their own thinking. As students learn to rely on their sense making and reasoning skills, they become better situated to build new, meaningful knowledge. Thus students learn that errors in thinking are not bad; rather, they are the stepping stones to increased understanding. The findings from this analysis suggest that by allowing students to work together in solving mathematical tasks, with minimal teacher assistance or intervention, and in an environment that provides the security

needed to participate in open discussions of their ideas, regardless of their mathematical validity, students can learn to reason, can improve their understanding of basic concepts, and can be empowered to exercise agency over their own knowledge of mathematics.

## References

- Arbaugh, F., Brown, C., Lynch, K., & McGraw, R. (2004). Students' ability to construct responses (1992-2000): Findings from short and extended constructed-response items. In P. Kloosterman & F.K. Lester (Ed.s), *Results and Interpretations of the 1990-2000 Mathematics Assessments of the National Assessment of Educational Progress* (pp. 337-363). Reston, VA: NCTM.
- Ball, D.L., & Bass, H. (2003). Making mathematics reasonable in school. In J.Kilpatrick, W.G.Martin, and D. Schifter (Eds.), *A Research companion to the Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: NCTM.
- Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A.E. Kristin & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Erlbaum.
- Cobb, P., Wood, T., & Yackel, E. (1992). Learning and interaction in classroom situations. *Educational Studies in Mathematics*, 23, 99-122.
- Davis, R.B. (1992). Understanding "Understanding". *Journal of Mathematical Behavior*, 11, 225-241.
- Davis, R.B. (1994). What Mathematics Should Students Learn? *Journal of Mathematical Behavior*, 13, 3-33.
- Davis, R. B. & Maher, C. A. (1993). What are the issues? In R. B. Davis & C. A. Maher (Eds.), *Schools, Mathematics, and the World of Reality* (pp. 9-34). Boston: Allyn and Bacon.
- Davis, R. B. & Maher, C. A. (1997). How students think: The role of representations. In L. D.English (Ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images* (pp. 93-115), Hillsdale, NJ: Lawrence Erlbaum Associates.
- Erlwanger, S.H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, 1(2), 7-26.
- Francisco, J.M. & Maher, C.A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, 24, 361-372.
- Hanna, G (1991). Mathematical Proof. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Kluwer, Dordrecht.
- Hanna, G. (2000). A critical examination of three factors in the decline of proof. *Interchange*, 31(1), 21-33.
- Hanna, G., & Jahnke, H. N. (1996). Proof and Proving. In Bishop, A., et al. (Eds.), *International Handbook of Mathematics Education*, Dordrecht: Kluwer Academic Publishers.

- Kamii, C. & Dominick, A. (1998). The harmful effects of algorithms in grades 1-4. In L. J. Morrow & M. J. Kenney (Eds), *The teaching and learning of algorithms in school mathematics: 1998 yearbook*. (pp.130-140). Reston, VA: The National Council of Teachers of Mathematics.
- Kamii, C. & Warrington, M.A. (1999). Teaching fractions: Fostering children's own reasoning. In L.V. Stiff & F.R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12 NCTM 1999 Yearbook* (pp82-92). Reston, VA: National Council of Teachers of Mathematics.
- Kouba, V.L. & Wearne, D. (2000). Whole number properties and operations. In E.A. Silver & P.A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*, (pp. 141-163). Reston, VA: NCTM.
- Mack, N. K. (1990). Learning fractions with understanding: Building on informal Knowledge. *Journal for Research in Mathematics Education*, 21, 16-32.
- Maher, C.A. (2002). How students structure their own investigations and educate us: What we have learned from a fourteen year study. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-sixth Annual Meeting of the International Group for the Psychology of Mathematics Education (PME26)* (Vol. 1, pp.31-46). Norwich, England:
- Maher, C.A. (2005). How students structure their investigations and learn mathematics: Insights from a long-term study. *Journal of Mathematical Behavior*, 24(1), 1-14.
- Maher, C.A. & Davis, R.B. (1995). Children's exploration leading to proof. Mathematical Science Institute of London, University of London, 87-105.
- Maher, C.A. & Martino, A.M. (1996). The development of the idea of mathematical proof: a 5-year study. In F. Lester (Ed.), *Journal for Research in Mathematics Education*, 27(2), 194-214.
- Maher, C.A. & Martino, A.M. (2000). From patterns to theories: Conditions for conceptual change. *The Journal of Mathematical Behavior*, 19, 247-271.
- Malloy, C.E. (1999). Developing mathematical reasoning in the middle grades: Recognizing diversity. In L.V. Stiff & F.R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12 NCTM 1999 Yearbook* (pp.82-92). Reston, VA: National Council of Teachers of Mathematics.
- Martino, A.M. & Maher, C.A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior*, 18(1), 53-78.
- Polya, G. (1981). *Mathematical discovery*. New York: Wiley.
- Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, 22(4), 405-435.
- Silver, E.A., Alacaci, C. & Stylianou, D.A. (2000). Students' performance of extended constructed-response tasks. In E.A. Silver & P.A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*, (pp. 307-343). Reston, VA: NCTM.
- Skemp, R. R. (1971). *The Psychology of Learning Mathematics*. Penguin Books.

- Steencken, E.P. & Maher, C.A. (2003). Tracing fourth graders' learning of fractions: early episodes from a year-long teaching experiment. *The Journal of Mathematical Behavior* 22(2), pp. 113-132.
- Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38, 289-321.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Translated by Michael Cole, V. John-Steiner, S. Scribner, E. Souberman. Harvard University Press, Cambridge, MA.
- Wearne, D. & Kouba, V.L. (2000). Rational numbers. In E.A. Silver & P.A. Kenney (Eds.), *Results from the seventh mathematics assessment of the national assessment of educational progress*, (pp. 163-193). Reston, VA: NCTM.
- Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin. & D. Schifter. (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp 227-236). Reston, VA: NCTM.