Incoherence of a concept image and erroneous conclusions in the case of differentiability

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Abstract:

The level of the coherence of a concept image conveys how well the cognitive structure concerning the concept is organized. This study considers the relationship between deficiencies in the coherence of the concept image and erroneous conclusions in the case of differentiability. The study is based on an interview where the student made conclusions contradictory to the formal theory of mathematics. He used an erroneous method to study the differentiability of piecewise defined functions. This method became the key factor which maintained the internal coherence of the concept image. It made it possible to build a cognitive structure whose basis was erroneous.

Keywords: Cognitive structure, Coherence of a concept image, Concept image, Definition, Derivative, Differentiability, Erroneous conclusions, Mathematical reasoning, Representation

1. Introduction

During the academic year 2004-2005, a grand total of 146 subject-teacher students in mathematics from six universities in Finland participated in a written test. Typically, 150-250 subject-teacher students in mathematics graduate in Finland annually. Most of the participants were at the final phase of their studies. In addition, 20 subject-teacher students from one university in Sweden participated in the test. The test contained a task where the students had to determine which of the given functions (both the graphs and the formulas were given) were continuous and/or differentiable. One of the functions was the function f_3 in Figure 1.

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$$f_3(x) = \begin{cases} x^2 - 4x + 3, & x \neq 4, \\ 1, & x = 4. \end{cases}$$

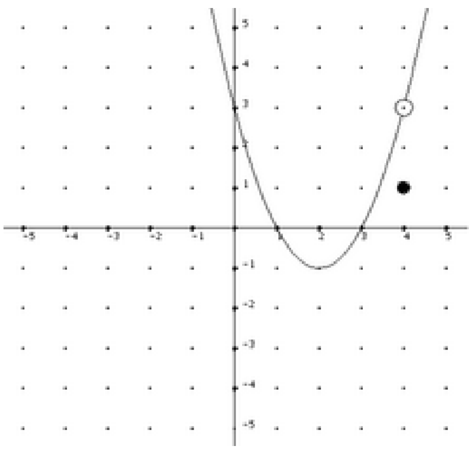

Figure 1: The graph and the formula of the function f_3 used in the test.

The starting point of the study presented in this paper was the observation that 38 Finnish and four Swedish students answered that the function f_3 was differentiable but not continuous. This outcome was very surprising and demanded an explanation. All of the students had during their studies encountered a theorem of calculus according to which continuity is a necessary but not sufficient condition for differentiability.

This study is based on the analysis of an interview of a student who had in the test answered that the function f_3 is not continuous but differentiable. In the interview, this student made several other erroneous conclusions. The goal of the study is to analyse how the conclusions in this interview were created and to discuss why some of them were erroneous with respect to the formal theory of mathematics. Several deficiencies were found in the knowledge structure concerning the concepts of derivative and differentiability. In the analysis, the theory about the concept image is applied. The term “coherence of a concept image” refers to the internal organisation of the knowledge structure concerning a certain concept. In order to clarify this term, a list of characteristics of a highly coherent concept image is presented in Section 2.2. In the analysis the coherence of the interviewee’s concept image is evaluated on the basis of these characteristics.

In this paper, the term “erroneous conclusion” stands for a result of a concluding process which is in contradiction with the formal theory of mathematics. It does not primarily refer to the illogicality of the concluding process. “Resulting misconception” could be an alternative expression for it.

2. Concept image and its coherence

2.1. Structure of the concept image

Tall and Vinner have defined the term *concept image* to describe the total cognitive structure that is associated with a concept (Tall & Vinner, 1981). According to them, a concept image includes all the mental pictures and associated properties and processes relating to the concept, and it is built up through experience during one's lifetime. For clarity, it is reasonable to define that every concept has only one concept image in an individual's mind. Different portions of the concept image can be activated in different situations, but as a whole, the concept image of one concept is an entity.

Tall and Vinner have defined the *concept definition* to be a form of words used to specify the concept (ibid). The concept definition generates its own concept image, which Tall and Vinner call the *concept definition image*. They have also separated a *personal concept definition* from a *formal concept definition;* the former means an individual's personal way to define the concept in practice, whereas the latter is part of the formal axiomatic system of mathematics. This system consists of axioms, definitions, undefined elementary concepts (e.g., a point and a line in geometry), rules of logic, and mathematical language, and it forms an *institutionalized way of understanding mathematics* (Harel, in press).

In order to understand the formal concept definition, which is presented, for instance, in a textbook or in a lecture, an individual has to interpret the expression in the definition: He/She has to create a *personal interpretation of the definition*. These interpretations are essential factors in the *personal way of understanding* mathematics (Harel, in press). The formal concept definition is usually unambiguous, but the personal interpretations of the definition may vary between individuals, and they may also depend on the context (Pinto, 1998; Pinto & Tall, 1999; Pinto & Tall, 2002). The personal interpretation of the definition does not mean the same as the personal concept definition: The latter is not necessarily based on the formal definition at all. It can be thought that every time when an individual applies a formal definition in a reasoning, he/she in fact applies his/her personal interpretation of the definition. According to the definition of Tall and Vinner, the concept definition image includes the total cognitive structure that is associated with the concept definition. Thus, it is very natural to think that the personal interpretation of the concept definition is part of the concept definition image, which, for one, is part of the whole concept image. A diagram describing the internal structure of the concept image is presented in Figure 2. The personal concept definition is not included in this diagram, because its location is not unambiguous: It may be equal with the personal interpretation of the formal concept definition, but it may also lie outside the concept definition image. In the latter case the individual's own way to define the concept is not based on the formal definition.

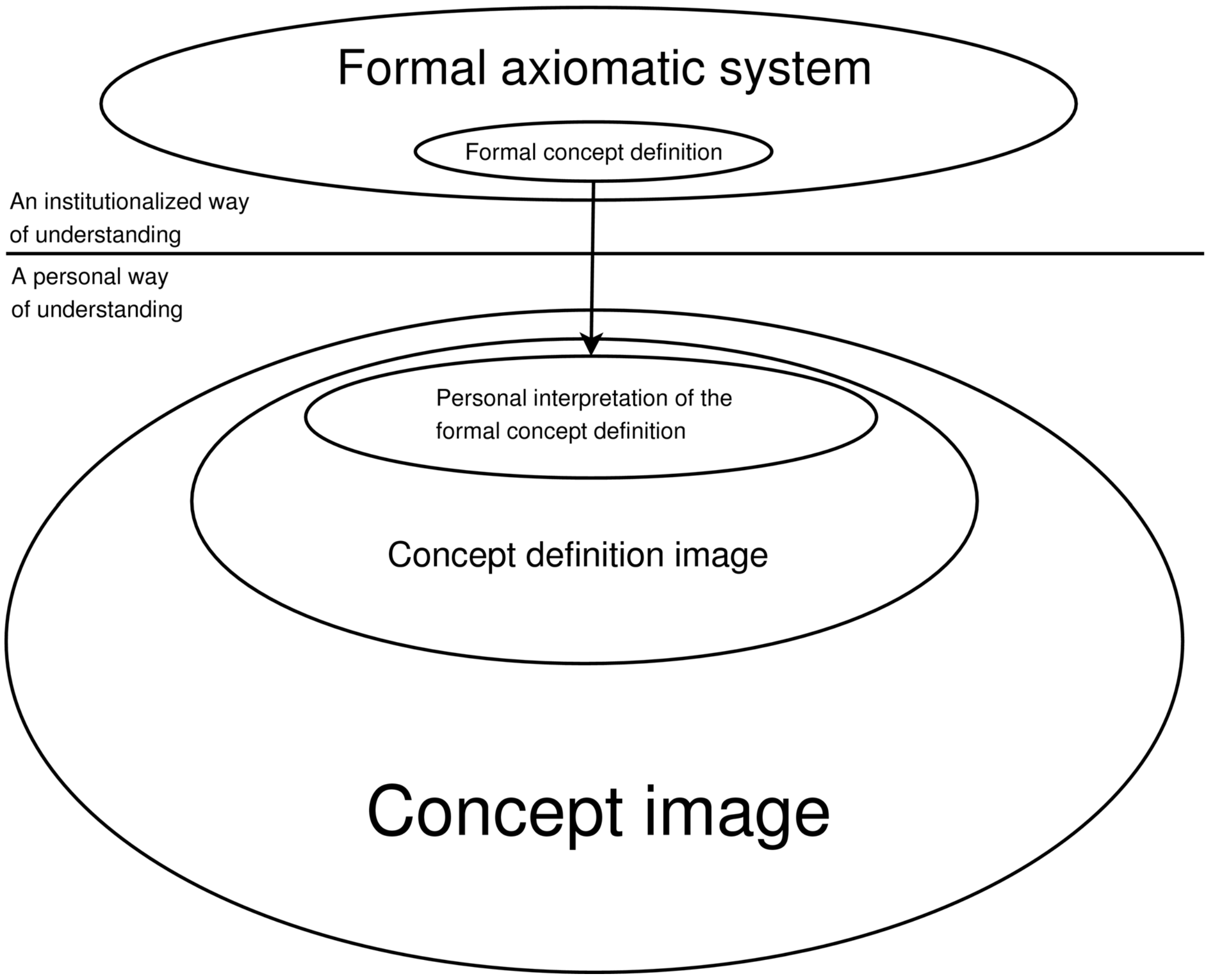

Figure 2: A diagram about the relationship between the concept image and the concept definition.

One reason for creating the terms concept image and concept definition was to separate reasoning based on the definition from reasoning based on other conceptions, representations and mental images. Thus the concept image and the concept definition were originally seen as opposite entities. In this way these terms are used, for example, in Tall's and Vinner's original paper (1981) and in Vinner (1991). Later Tall (2003; 2005) has reported about differences between him and Vinner regarding the view about the location of the concept definition: Tall considers the concept definition rather as a part of the concept image whereas Vinner has emphasized the distinction of them. In my model the formal concept definition is located outside the concept image, but through its personal interpretation it has a notable effect on the concept image.

My model may seem to have a positivistic or naturalistic basis: There is a system outside the mind of human which one attempts to understood. However, this model does not describe the whole process of learning or thinking, but only the role of the definition in the concept image and the process of understanding a given definition. The model in this form is not suitable for situations where the definition is not static but it is created or reconstructed.

2.2. Coherence of a concept image

Creative mathematical thinking requires that the concept image includes a variety of multifaceted conceptions, representations and mental images concerning the meaning and properties of the concept and relationships to other concepts. Representations and mental images[2] may be, for example, verbal, symbolic, visual, spatial or kinesthetic. It is also important for the concept image to be well ordered. The term *coherence of a concept image* is used to refer to the level of organization of various elements in the concept image. In practice, the concept images are hardly ever fully coherent or fully incoherent, but their level of coherence varies. To clarify the term, some criteria for a high level of coherence of a concept image are presented in the following list:

1. An individual has a clear conception about the concept.
2. All conceptions, cognitive representations and mental images concerning the concept are connected to each other.
3. A concept image does not include internal contradictions, like contradictory conceptions about the concept.
4. A concept image does not include conceptions which are in contradiction with the formal axiomatic system of mathematics.

Like the structure of a concept image as a whole, the level of coherence is not static but it changes all the time during mental activities concerning the concept.

The way the person views the concept (cf. criterion 1) may vary depending on the context. However, usually one of these conceptions is above the others, and it can thus be considered, according to Tall's and Vinner's terminology, as a personal concept definition. If the concept image is highly coherent, the different conceptions are mentally connected to each other (cf. criterion 2), but they are not in contradiction with each other (cf. criterion 3) or with the formal theory of mathematics (cf. criterion 4).

The criterion 2 means that there exist mental connections between the elements of the concept image. Goldin and Kaput (1996) have defined that the connection between two representations is *weak* if an individual is able to predict, identify or produce one representation from the other, and the connection is *strong* if an individual is, from a given action upon one representation, able to predict, identify or produce the results of the corresponding action on the other representation. Hähkiöniemi (2006a, 2006b) has defined that a person makes an *associative* connection between two representations if he or she changes from one representation to another and that a person makes a *reflective* connection between two representations if he or she uses one representation to explain the other. These are examples of potential types of the connections between the elements of the concept image. A highly coherent concept image makes possible both weak and strong connections (cf. Goldin & Kaput) and, respectively, associative as well as reflective connections (cf. Hähkiöniemi) between representations concerning the concept. Observed strong and reflective connections can be considered as stronger indications of the coherence than weak and associative connections.

For example, the concept of the derivative is according to its formal definition a limit of a difference quotient. On the other hand, the derivative can be visually interpreted as a slope of the

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2 These both are widely used terms in the discipline of mathematics education, but their meaning is not unambiguous. In this connection these (both) are considered as mental configurations which represent (corresponds, associates, stands for symbolizes etc.) something else. This view is in accordance with Goldin's and Kaput's (1996) traditional definitions for the concept of representation.

tangent line, or it can be understood as a measure of an instantaneous rate of change. These are three different interpretations concerning the meaning of the concept of the derivative. If an individual has a highly coherent concept image, he/she is able to utilize all these interpretations in a problem solving process regardless the original form of the problem. If needed, he/she is able to change interpretation (weak or associative connection), and if changes in a system based on one interpretation happen, he/she is able to see the corresponding changes in another system which is based on another interpretation (strong connection). He/she is also able to explain, for example, on the basis of the definition why it is justified to consider the derivative as a slope of the tangent line (reflective connection).

The connections between elements of the concept image are important for preventing internal contradictions (cf. criterion 3). For preventing contradiction with the formal theory (cf. criterion 4), it is important that the elements of the concept image have connections also to the formal axiomatic system. This requires that the personal interpretation of the formal definition is correct and it has a central role in the concept images: Other conceptions, representations and mental images concerning the concept should be *reflectively connected* (cf. terminology by Hähkiöniemi) to this interpretation, in other words, they should be justifiable on the basis of this interpretation.

If the coherence of a concept image has some deficiencies with respect to criterion 3, it is very probable that it has deficiencies also with respect to criterion 4, because the formal axiomatic system of mathematics is (at least it should be) consistent[3]. On the other hand, it is possible that a concept image includes entities which are internally coherent, but which are in contradiction with the formal theory. These kind of entities may be based, for instance, on one or more misconception, misinterpretation or erroneous conclusion.

To some extent, the coherence of concept images and the *conceptual knowledge* mean the same thing. The term conceptual knowledge has been defined as a knowledge of relationships between pieces of information (Hiebert & Lefevre, 1986) or as a knowledge of particular networks and a skilful drive along them (Haapasalo & Kadijevich, 2000). The network consists of elements (concepts, rules, problems, and so on) given in various representation forms (ibid.). Thus, if a concept image of a certain concept includes a variety of knowledge and the structure is highly coherent, the level of the conceptual knowledge regarding this concept can be considered to be high. However, the high level of conceptual knowledge in a broader sense provides rich connections between the concept images of various concepts. It can be assumed that the conceptual knowledge, on one hand, provides networks consisting of connections between elements of knowledge inside each concept image and, on the other hand, a network consisting of connections between various concept images. However, the theories about concept image are not very usable in analyzing the whole structure of knowledge, but they are useful when concentrating on the knowledge concerning one concept at a time. In the latter case the conceptions about the relationships between a concept under analysis and other concepts can be considered as elements inside the concept image of the concept under analysis. For example, let us assume that an individual has a conception that continuity is a necessary but not sufficient condition for differentiability. When analyzing the knowledge structure of differentiability, this conception can be considered as an element of the concept image of differentiability, and, respectively, it can also be seen as an element of the concept image of continuity when concentrating on continuity. So this element, which is common for both concept images, forms a link between the

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3 According to Gödel's incompleteness theorem, proven in 1930, an axiomatic system cannot be proven consistent. However, the consistency is a fundamental goal in building the theory of mathematics.

concept images.

The level of coherence of a concept image can be examined by finding from an individual's behavior indications referring either to coherence or incoherence of the concept image. In the analysis presented in this paper, we have mainly concentrated on finding indications of incoherence of an interviewee's concept image of differentiability.

2.3. Previous studies

Several studies concerning mathematics students' reasoning have shown significant indications of a low level of coherence of concept images. In the following, some studies concerning this in the area of calculus or basic analysis are briefly reviewed. In some cases, the results of the study are related to the above criteria of the coherence of a concept image.

With respect to the limit concept, Juter (2005) has shown that students can have contradictory conceptions about the attainability of a limit value of a function so that conceptions which come up in a theoretical discussion differ from conceptions used in problem solving situations. For instance, some students in Juter's study said in a theoretical discussion that a function cannot attain its limit values, but in a problem solving situation they considered it to be possible. This indicates contradictory conceptions inside the concept image (cf. criterion 3) and with respect to the formal theory (cf. criterion 4). One reason for the conception about the unattainability of the limit was an erroneous interpretation of the definition of the limit.

Zandieh's study (1998) considered high-achieving high school students' abilities to relate the formal definition of the derivative to other aspects of their understanding. The results varied between students, and, according to Zandieh, the crucial factors in this were ability to understand mathematical objects and processes also in other contexts than in the symbolic one and, on the other hand, ability to use mathematical symbols as a language to express knowledge in other contexts. This result indicates the importance of connections inside the concept image (cf. criterion 2).

Aspinwall et. al. (1997) have shown how, in the case of the derivative, an uncontrollable use of visual images may become a source of conflicts. In their study a student reasoned on the basis of a graph that a parabola presenting the function x^2 approaches asymptotically a vertical straight line when x increases or decreases enough. On the other hand, he reasoned that the graph of the derivative of this function is a straight line. He regarded these conclusions contradictory to each other. According to the interpretation of Aspinwall et. al., this conflict was caused by the inadequate control in using the visual image. It can also be interpreted that the connections inside the concept images of the function and the derivative were inadequate. A more thorough consideration of connections between the graph and symbolic expression of the function might have made the needed control possible and thus prevented the erroneous conclusion about the asymptotic approach.

Aspinwall and Miller (2001) have discussed possible methods to explore and to improve the coherence of the concept image in the case of the derivative. They have analyzed possible conflicts concerning, for example, the interpretation of the derivative as the slope of the tangent and the relationship between the average and instantaneous rate of change. They have also studied the coherence of the concept image in the case the concept of definite integrals.

It is natural to assume that, compared to students, mathematicians have more coherent concept images of mathematical concepts and thus a more coherent knowledge structure with respect to mathematics as a whole. Some studies have considered this issue. For instance, according to Raman

(2002; 2003), mathematicians are able to see connections, *the key idea*, between heuristic ideas and formal proofs, but students consider these arguments separately, without seeing the connections. Stylianou (2002) has shown that mathematicians, during problem solving processes, very systematically take turns between visual and analytical steps, but many students cannot utilize visual representations in analytical problems at all. These findings have a significant role in explaining the differences in performances between mathematicians and students.

Even though concepts are determined by definitions in the formal axiomatic system, in students' concept images they tend to stay as isolated cells. It has been shown that students, in connection with basic analysis, have difficulties in consulting definitions and that they often avoid using them (Cornu, 1991; Pinto, 1998; Vinner, 1991). This may be an essential problem causing incoherence of a concept image. Pinto's study (1998) revealed that students also have different modes to work with definitions and to deal with a formal theory: *Formal thinkers* attempt to base their reasoning on the definitions, while *natural thinkers* reconstruct new knowledge from their whole concept image (Pinto & Tall, 2001). Both modes have their own advantages and disadvantages with respect to the coherence of a concept image: For instance, if the meaniong to the concept is extracted from the formal defintion, the concept image is well tied to the formal theory, but, on the other hand, the informal imagery may leave poorly connected. However, both modes can lead to a success or a failure (ibid; Pinto, 1998; Pinto & Tall, 1999).

3. Methodology

The student, whose interview is thoroughly analysed in this paper, was selected among eight interviewed students, who had in the written test answered that the function f_3 in Figure 1 was differentiable but discontinuous. This particular interview seemed to offer usable data concerning concluding processes, erroneous conclusions, and coherence of the concept image. The interviewee, called Mark in this paper, was majoring in mathematics. He had studied five years at university, and, according to his own estimate, his success in studies had been on the average level. Mark told that in the future he would like to teach mathematics, by choice, at a lower secondary school.

The main goal of the interviews was to study the students' conceptions about the meaning of the derivative and differentiability and their abilities to understand relationships between the formal definitions and some visual interpretations of these concepts. In the interview, the interviewee was asked to justify the differentiability or nondifferentiability of the functions presented in Figure 3. In some cases continuity was also considered. The functions were given to the interviewee by showing both the symbolic expression of the formulas and the graphs on paper. The interview also included a discussion about the visual meaning of the derivative and differentiability and the relationships between continuity and differentiability.

The functions f_1, f_2 and f_3 were used also in a task of the written test. In this task, the students were asked to determine which of the functions were continuous and which of them were differentiable.

$$f_1(x) = \begin{cases} x+1, & x < 1, \\ -2x+6, & x \geqslant 1. \end{cases} \qquad f_3(x) = \begin{cases} x^2-4x+3, & x \neq 4, \\ 1, & x = 4. \end{cases}$$

$$f_2(x) = \begin{cases} x+2, & x < 1, \\ -2x+5, & x \geqslant 1. \end{cases} \qquad f_4(x) = \begin{cases} x, & x < 1, \\ x+1, & x \geqslant 1. \end{cases}$$

Figure 3: The formulas of the functions used in the interview.

The interviews were semistructured: The main questions were planned in advance, and many additional questions emerged during the interview. The formal definitions of continuity, derivative and differentiability were given to the participants both in the written test and in the interview. The forms of the given definitions are presented in Figure 4. The only tools allowed in the interview were pen and paper. The interviews were videotaped so that the video camera was focused on the paper.

Definitions

Continuity
A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $x_0 \in \mathbb{R}$, if and only if the limit $\lim_{x\to x_0} f(x)$ exists and

$$\lim_{x\to x_0} f(x) = f(x_0).$$

A function is *continuous*, if and only if it is continuous at all points in the domain of the function.

Derivative and differentiability
A function $f : \mathbb{R} \to \mathbb{R}$ has a *derivative* or it is *differentiable at a point* $x_0 \in \mathbb{R}$, if and only if the limit

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. Then the derivative of the function f at the point x_0 is equal to the value of this limit.

A function is *differentiable*, if and only if it is differentiable at all points in the domain of the function.

Figure 4: The definitions of continuity, derivative and differentiability given to the participants in the written test and in the interview (translated from Finnish).

The thorough analysis of Mark's interview mainly applied the principles of the video data analysis procedures presented by Powell et al. (2003). The interview was first transcribed from the video. Then, the transcribed data was divided into episodes so that every episode included reasoning concerning one question, for example, the question of differentiability of one of the presented functions or the question concerning the relationship between differentiability and continuity. After that, the reasoning during the episodes was described and *critical events* (ibid.) with respect to the progress of the reasoning were identified. By comparing episodes and critical events of the interview and by searching common features between them, it was possible to find some features of the interviewee's thinking which were typical during the whole interview.

4. Results concerning erroneous conclusions and the structure of the concept image

4.1. A fundamental change of view

Mark's conceptions about the relationship between continuity and differentiability changed due to the conclusions which he made during the interview. Unfortunately, the change happened to a more erroneous direction.

At the beginning of the interview Mark believed that differentiability presumed continuity. In fact, continuity was the first property which Mark mentioned when the interviewer asked him about the meaning of differentiability (see excerpts 10-11 in Section 4.2.). However, during the interview, Mark reasoned that the function f_4 was differentiable but not continuous. This forced him to change his view, although it was not too easy for him.

(1) Interviewer: Then, how about the question whether differentiability presumes continuity? How do you respond to that?

(2) Mark: If I now here claim that this function is differentiable, it means that it does not presume continuity.

(3) Interviewer: You had a memory that it would presume.

(4) Mark: Yes, I did. This appears to be contradictory…

[...]

(5) Mark: This time, I say that it does not presume continuity!

Observable erroneous conclusions –as well as erroneous conceptions– with respect to the formal theory concerning some concept can be regarded as indicators of the incoherence of the concept image of that concept. Thus, the above-presented conclusion as such reveals that something was wrong with respect to the coherence of the concept image of differentiability. However, in the following analysis the viewpoint is in the opposite direction: The goal is to analyse issues relating to the coherence of the concept image and from this basis to discuss how the above erroneous conclusion was built up. The main attention is focused on Mark's study about the differentiability of the functions f_1–f_4. This process is described in Section 4.3. In Section 4.2 some indications of incoherence of the concept image of differentiability, which came out before Mark started this process, are described.

4.2. Indications of incoherence of the concept image before the study of differentiability of the given functions

The discussion with Mark about the meaning of continuity and differentiability revealed some indications of incoherence in his concept images of these concepts. The clearest indication was his explicit uncertainty about whether the cornerlessness of a graph was a property of continuity or a property of differentiability. This came out when the interviewer asked Mark to explain what the continuity of a function means in practice. Mark began to think about properties of a continuous function:

(6) Mark: A connected graph, a graph of a function which does not jump and... There are no sharp corners... Or does this belong to differentiability?

Mark seemed to be unsure if it is possible to have corners in a graph of a continuous function. At first he guessed that it is not possible:

(7) Mark: There can't be, I guess.

Then Mark tried to argue this by characterizing the continuity on the basis of his subjective, everyday life associations of the word continue:

(8) Interviewer: Why not?

(9) Mark: Why not? It does not continue then. If you drive a car suddenly to a sharp corner, then... it seems not to continue.

Mark did not seem to have a particularly clear conception about the meaning of differentiability, either. First, it was difficult for him to mention any other property of a differentiable function than continuity:

(10) Interviewer: What kind of function is differentiable? What should it be like in order to be differentiable?

(11) Mark: Continuous.

(12) Interviewer: Continuous? Does it have to be something else?

(13) Mark: Can I resort to the definition? If it helped me in some way...

However, Mark did not say anything explicit about the definition. Finally, he mentioned cornerlessness:

(14) Mark: There cannot be (in the graph of a differentiable function) these [...] corners, because we cannot draw a tangent to a sharp corner. Or, in fact, we can draw the tangent almost anyhow we like.

After that the discussion moved to continuity of function f_2. In this discussion Mark changed his view regarding continuity and cornerlessness. In the written test Mark had answered that the function f_2, whose graph includes a corner, is continuous. This is against the view that he presented above (cf. excerpts 6-9). When the interviewer presented this answer to him, he was ready, again using his subjective associations of the word continue, to argue that a graph of a continuous function can include corners:

(15) Mark: It does not break the function, its graph. If we look at this, we can see that it continues. (*He traces the graph with a pen.)*

Above presented hesitances show that in the beginning of the interview Mark did not have very clear conception about the meaning of continuity and differentiability (cf. criteria 1).

Another contradiction between test answers and the conceptions which came out in the interview was following: In the test Mark had answered that function f_3 was discontinuous but differentiable, whereas in the interview Mark considered continuity as a prerequisite of differentiability (cf. excerpts 10-11). The interviewer asked Mark to explain why he had thought function f_3 to be differentiable:

(16) Mark: At the point four the derivative is zero.

(17) Interviewer: Why?

(18) Mark: Because it (value of the function) is a constant!

This excerpt shows that Mark was aware that the derivative of a constant function is zero and that differentiability means the existence of the derivative. However, he applied these facts in an erroneous way for the function f_3, assuming that the above-presented reasoning was really his argument for differentiability during the test. It can be interpreted so that, in the test situation, these pieces of knowledge had erroneous connections between them in the concept image (cf. criterion 2), and due to this Mark considered the rule concerning the derivative of a constant function to be applicable. Of course, it cannot be claimed that Mark did not know the prerequisites of this rule, but at least in this situation he did not take them into account correctly.

As a whole, the above-presented observations reveal that significant deficiencies with respect to coherence appeared in Mark's concept image of differentiability at the beginning of the interview, before he began to study the differentiability of the functions f_1–f_4. The observed deficiencies mainly concerned the meaning of the concept and the connections between elements of knowledge inside the concept image. However, it has to be noticed that the deficiencies might not have been permanent: The structure of the concept image may have changed already in the situations where the deficiencies appeared.

4.3. Vitality of a method based on the differentiation rules

When solving problems concerning differentiability of piecewise defined functions, Mark in several cases first differentiated both expressions used in the definition of the function by using differentiation rules, and then checked if both expressions obtained an equal value at the point where the expression is changed.

Mark was told to begin by considering the differentiability of the function f_2. First, Mark explained visually, by using tangents, why this function was not differentiable at the point x=1. He explained that it was not possible to draw an unambiguous tangent line at the corner. Then the interviewer asked Mark to calculate the right-hand and left-hand limits for the difference quotient of the function f_2 at the point x=1, when *h* in the definition of the derivative approaches 0. Mark differentiated the expressions x+2 and -2x+5 and gave the answers 1 and -2. He said:

(19) Mark: The use of the difference quotient would lead to the same result.

Furthermore, the interviewer asked him to calculate this by using the definition. Mark calculated the limit of the difference quotient for the function x+2 and came up with 1:

(20) Mark: *h* is negative. [...] It becomes *h*/*h,* and it is one. And if *h* approaches zero...

In this calculation Mark did not specify the point x=1 but performed the calculation generally for the function x+2 at a point $x=x_0$. (See Figure 5.) In fact, this is not a correct way to calculate the left-hand limit at the point x=1, because according to the definition of the function f_2 the expression x+2 is not in force at this point.

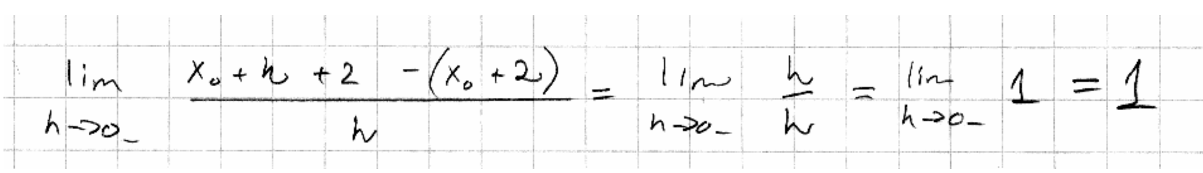

Figure 5: Mark's way to use the definition of the derivative in calculating the left-hand limit of the difference quotient of the function f_2 at the point x=1.

After that he said that the right-hand limit of the difference quotient could be found similarly.

In this way Mark introduced his differentiation method to study differentiability. Excerpt 19 reveals that already before the use of the definition Mark had a clear view that his method was compatible with the definition. It does not explicitly come out from the data why he believed so, but a possible explanation could be that very often in practice, especially at high school, the use of the definition of the derivative is replaced by the differentiation rules. This perhaps was the origin of Mark's differentiating method. The calculation with the definition offered Mark an additional confirmation for the correctness of his method.

For the function f_1, Mark applied the same method, and it seemed to work very well:

(21) Mark: This is a similar situation... at x=1... Let's consider the lower part *(refers to the expression)* first. The derivative is minus two for the lower part, and then, if we try when x<1, the derivative approaches one, or it equals one.

Because the derivatives of the expressions x+1 and -2x+6 were not equal, Mark's conclusion was that the function f_1 was not differentiable.

As mentioned above, in the written test Mark had answered that the function f_3 was differentiable (cf. excerpts 16-18). However, in the interview he wanted to apply his differentiation method also to this function. Thus he differentiated the expressions x^2-4x+3 and 1 and came up with the expressions 2x-4 and 0. Because these did not take the equal value at the point x=4, Mark concluded that the function f_3 was not differentiable. He rejected his previous conclusion and believed that the conclusion obtained by applying the differentiation method was the right one. He commented:

(22) Mark: My view changes when I think over these things more and more.

In this situation Mark met an obvious conflict. He had used two different methods for examining the differentiability of the function f_3, and these methods led to opposite conclusions. In fact, according to the formal theory, both methods were erroneous. Mark, however, believed that the result obtained by the differentiation method was the right one, and he was ready to reject the result obtained by the other method. This can be interpreted as a sign of confidence in the differentiation method: At least, it shows that in this situation his confidence in it was stronger than in the other method. However, Mark did not consider what was wrong in the other method, and thus, this part of the conflict was left unsolved.

The question about the differentiability of the function f_4 was hard for Mark. As before, he started by calculating the derivatives of the expressions used in the definition of the function (the expressions x and x+1) and noticed that these were equal at the point x=1. It is notable that when calculating these, he spoke about the difference quotients:

(23) Mark: The difference quotients are equal in both domains.

This again suggests that Mark believed that the use of his differentiation method could be substituted for the explicit use of the definition of the derivative. According to Mark, the result forced the function f_4 to be differentiable. However, Mark immediately saw from the graph that the function f_4 was not continuous and he remembered very clearly that continuity is a necessary condition for differentiability. This caused a serious conflict for Mark, but in spite of that, his confidence in his method was strong:

(24) Mark: Yes, both derivatives are equal if we come either from left or from right. [...] If we think only that they are equal... then it has to be differentiable... But it is not continuous at that point!

In this excerpt, Mark spoke about the equality of "both derivatives", even though, according to the formal theory, differentiability requires the equality of the both-hand limits of the difference quotient. This indicates a confusion between concepts: Perhaps Mark was not able to recognize the difference between the limit of the derivative and the limit of the difference quotient in this situation.

Then Mark began to doubt his memory. He tried to find another differentiable but discontinuous function. He wondered if the tangent-function (the function *f(x) = tan x*) could be one example. Finally, he decided to explore the differentiability of the function f_4 by using the definition of the derivative explicitly. However, he made a mistake in this: He calculated the difference quotients generally at the point $x=x_0$ and separately for the expressions x and x+1 (see Figure 6). This is not the correct method to study differentiability at a point where the defining expression of the function changes. Using the terminology presented in Section 2.1, it can be said that Mark's personal interpretation of the formal concept definition was not consistent with the formal theory in this situation.

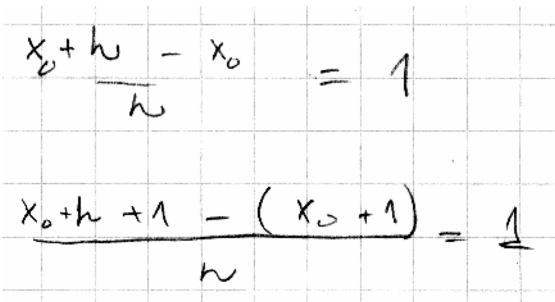

Figure 6: Mark's way to calculate the left-hand and right-hand derivatives of the function f_4 at the point $x=1$.

Because Mark got equal results from both of these calculations, he concluded that the function f_4 was differentiable:

(25) Mark: One comes from both. It could be reasoned from this that it is differentiable.

(26) Interviewer: Is this your answer?

(27) Mark: Ok, let it be my answer, this time!

Finally, he was ready to break his strong conception that continuity is a necessary condition for differentiability (cf. excerpts 1-5).

Like with the function f_3, Mark again met an obvious conflict with function f_4. Now there were, against each other, his very strong memory that differentiability presumes continuity and the result based on the differentiation method, on which he had relied in the three previous cases. In the case of the function f_3, it was not difficult for Mark to reject the other conclusion, but in the case of function f_4 he felt he could not reject either of the results, even if they were contradictory. In the latter case he was convinced of both results. It seems that the explicit -indeed, erroneous (cf. Figure 6)- use of the definition had a crucial role in the solution of this conflict, but, furthermore, after using the definition, it was not easy for Mark to reject his memory concerning the continuity of a differentiable function (cf. excerpts 25-27 and 1-5).

After the interview, the interviewer gave a brief feedback for Mark about his performance. He, among others, revealed that continuity is a necessary condition for differentiability. In this situation there was not time for an extensive discussion, and Mark's reactions for the feedback were not taped.

5. Discussion

Above we have presented how Mark's interview revealed several indications of the incoherence of his concept image of differentiability, and described his process to study the differentiability of the given four functions. During this process, Mark made many erroneous conclusions. In the following we will discuss which matters can be learnt about mathematical thinking and learning through this analysis which could help us to improve teaching practises.

The above analysis reveals several cases in which **an erroneous conclusion was a consequence of an erroneous way to connect the pieces of knowledge**. In these situations single pieces of knowledge, as such, were correct, but they were connected in an erroneous way. This suggests that the knowledge about relationships is deficient. The explanation of the test answer regarding differentiability of function f_3 (cf. excerpts 16-18) is an illustrating single example of erroneous connections. Mark's differentiation method, also, can be seen to be based on erroneous connections between pieces of knowledge concerning existence of the derivative and differentiation rules.

This analysis shows also that **misconceptions and erroneous conclusions may lead to cognitive structures, which are, at least in some extent, internally coherent, but whose basis is erroneous**. In this study, Mark constructed a structure which was based on the conception that differentiability of a piecewise defined function can be studied by checking if derivatives of the expressions used in the definition of the function obtain an equal value at the point where the expression is changed. This method seemed to work very well in the cases of the functions f_1 and f_2, and in the cases of the functions f_3 and f_4 he rejected results which were in contradiction with it. Mark also became convinced that this method was compatible with the formal definition. With a strong confidence on this method, Mark changed his previous conception about the continuity of a differentiable function. In this way the differentiation method became a key factor for the internal coherence of the concept image of differentiability. Mark reconciled the other conceptions and results with the differentiation method, and the confidence in it maintained the internal coherence of the concept image. In fact, it would have been interesting to continue discussion by revealing for Mark that differentiability really presumes continuity but not giving any other feedback in this phase. This would have broken the internal coherence, and an extensive reconstruction would have been needed to repair it. Therefore, this study also illustrates that **sometimes the fundamental reason for erroneous conceptions can lie deep in the knowledge structure**: The conception that a differentiable function can be discontinuous was a result of a quite extensive reasoning process which was based on an erroneous method to study differentiability of piecewise defined functions. In practice, discussion with other people, the use of literature or another kind of social interaction often influences the process of constructing the knowledge structure and prevents the development of very wide-ranging erroneous structures. This is one reason why the social interaction in its different forms is important in the learning of mathematics. It contribute to recognizing misconceptions even by judging some conceptions directly erroneous or by bringing out situations where conflicts might be created. In this way the misconceptions are probably recognized earlier than it may be happened in an individual study.

This study also brings out some issues about the role of the definition in constructing the concept image. First, **it is important that the personal interpretation of the formal concept definition is correct**. If Mark had used the definition of the derivative in a correct way when he studied differentiability of the function f_4 (cf. Figure 6; excerpts 25-27), he would have met a conflict which could have forced him to re-examine his differentiation method. Already when calculating the left-hand-limit for the difference quotient in the case of the function f_2 (cf. Figure 5; excerpts 19-20), a more careful examination of the definition might have had a similar effect. Another conclusion which comes out from this study is that **the definition −or, in fact, its personal interpretation− should have a central role in the reasoning concerning the concept in question**. Reasoning concerning the concept should be based on the definition, or, at least, an individual should be aware why the reasoning is in accordance with the definition. In Mark's reasoning the definition appeared to have only a minor role. The only situation where Mark without the interviewer's intervention used the definition was the conflict in the case of the function f_4, but even then the definition was not the primary method to resolve the conflict. Thus, it seems that Mark had a tendency to avoid using the definition. Instead the definition, the differentiation method became a crucial criterion for the differentiability in his reasoning. When derivative is considered for the first time in mathematics education, for example, in upper secondary school, the definition is usually left to the background, and the use of the differentiation rules are emphasized. This may be one reason why Mark avoided the use of the definition. However, as discussed above, Mark probably believed that his method was compatible with the definition. This maybe the reason why he did not feel a need to use the definition. On the other hand, the data does not explicitly show whether Mark had understood the crucial role of the definition: Did he understand that the definition determines the final truth regarding the concept, or did he consider the definition only as one description of the meaning of the concept among others? The observation that the definition-based argument seemed to resolve the conflict in the case of the function f_4 (cf. Figure 6; excerpts 25-27) defends the former view.

The observations concerning students' study of differentiability made by Tsamir et. al. (2006) are quite similar to the results of this study. In their study, three prospective teachers were able to give a correct definition for the derivative, but, in spite of that, they did not use it in the problem solving, and they reached erroneous conclusions. When studying the differentiability of the absolute value function ($f(x)=|x|$), one of these students used the same kind of a method as Mark.

This study shows that there exists a notable interaction between the structure of the concept image and conclusions which are attained by reasoning. Single conclusions may have a wide-ranging influence on the structure of the concept image, and on the other hand, conclusions depend on the existing structure of the concept image. The list of criteria for coherence of a concept image offers a framework for analysing mathematical reasoning, especially reasoning concerning one concept. It could also be interesting to analyse a longer-term learning process by using this framework and in this way study how the coherence of a concept image is developed.

Almost every day mathematics teachers meet in their work erroneous conclusions made by students. Many of these are random careless mistakes, but others are based on a deliberate reasoning. In the latter case a careful personal discussion with a student may be needed in order to find out how deep in the knowledge structure the problems lie. In the learning process it is important that the pieces of knowledge which an individual learns and which he/she already knows form coherent entities. How could teachers and designers of textbooks and curricula take this goal into account? This study highlights two factors: First, the fundamental role of definition should be emphasized. Therefore, in teaching tasks in which the definition is really needed as a central part of reasoning should be used. Second, there should be tasks which lead students to critically reflect qualities of mathematical concepts and relationships between concepts. Especially, controlled conflict situations may offer fruitful starting points for this kind of reflection.

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