History of Mathematics in Mathematics Education: a Saussurean Perspective

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INTRODUCTION

It is not only because of a certain eclecticism in mathematics education research that semiotic ideas have begun to take root there: it is also because of the dawning recognition that key areas of interest in mathematics education genuinely have a semiotic nature. For this, one need only point to research focused on meaning, communication, language, and culture (*e.g.*, Presmeg, 1997; Ernest,1997; Radford, 2001; Brown, 2001). The ways in which semiotics informs cultural aspects of mathematics education, particularly those connected with the history of mathematics, were highlighted in a discussion session at the 2003 meeting of the International Group for the Psychology of Mathematics Education (PME27). The present paper continues the theme of the PME27 semiotics session by considering how semiotics can help clarify a problematic aspect of combining history of mathematics and mathematics education.

Before outlining that particular problem, I need to say in what corner of the semiotic field I plan to dwell. For semiotics has become a very large and many-colored thing. Indeed, the history of semiotics, the study of signs, can be traced quite far back into the past—as far back even as Classical times. At the end of the 17th century, Locke made the study of signs one of the three basic sciences "within the compass of human understanding," and he gave this study the name, *Semeoitica* (Locke, 1694, Book IV, chapter xx).[2] In the 18th and 19th centuries, figures (of equal or greater importance in the history of mathematics!) such as Lambert and Bolzano also made serious studies of semiotics (see Jakobson, 1980). But the figures we most associate with the modern study of semiotics are, surely, Charles Sanders Peirce (1839-1914) and Ferdinand de Saussure (1857-1913). Peirce's semiotic ideas have been quite influential, and, perhaps because of his exhaustive taxonomic approach to the subject, they have found broad application far beyond logic and linguistics. Saussure's semiological[3] ideas seem to have had somewhat less influence than Peirce's, even in

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[2] Locke describes the this science as follows: ...the third branch [of science] may be called *Semioitica*, or the *doctrine of signs*; the most usual whereof being words, it is aptly enough termed also logic: the business whereof is to consider the nature of signs, the mind makes use of for the understanding of things, or conveying its knowledge to others. For, since the things the mind contemplates are none of them, besides itself, present to the understanding, it is necessary that something else, as a sign or representation of the thing it considers, should be present to it: and these are ideas…(Locke, 1694, Book IV, chapter xx).

[3] Whereas Peirce called the general study of signs, *semiotic*, Saussure called it *sémiologie*—both, as was Locke's term, were based on the Greek word for 'sign' (which, incidentally, is also the word for 'point' in mathematical contexts) ⬧♒❍♏♓☐■.

linguistics where Saussure thought they should have their greatest impact.[4] In thinking about mathematics as a cultural system, however, Saussure's ideas have a cogency, which, to my mind, has not been sufficiently appreciated. Thus, in this paper, I will show how a specifically Saussurean approach to semiotics provides insight into the problem of history of mathematics and mathematics education.

The paper will comprise four parts. In the first part I shall present the theoretical difficulty in combining the history of mathematics and mathematics education. Next, I shall review some of Saussure's semiotic ideas, particularly, his notion of a sign and his notions of synchrony and diachrony. In the third part I shall show how these ideas are apposite to the difficulty presented in the first part. Finally, I shall consider how the semiological outlook discussed in the paper may serve as a foundation for a more humanistically oriented mathematics education.

1. THE PROBLEM OF HISTORY OF MATHEMATICS AND MATHEMATICS EDUCATION[5]

The main problem to be addressed in this section is whether there really is a problem at all in combining the history of mathematics and mathematics education. If there is a problem, most educators, I think it is fair to say, would consider it more practical than theoretical. In other words, what one has to worry about in combining history and mathematics education is chiefly how to do it: What examples should one chose for what material? What kind of history of mathematics activities can be incorporated into the ordinary mathematics curriculum? How does one find time for such activities? How does one find a place for history of mathematics in teacher training? Yet, when one follows these questions to their theoretical end, one begins to see a theoretical problem.

Take, for example, the problem of time, for this would appear the most practical of these practical problems. Avital (1995, p.7) says in this connection, "Teachers may ask 'Where do I find the time to teach history?' The best answer is: 'You do not need any extra time'. Just give an historical problem directly related to the topic you are teaching; tell where it comes from; and send the students to read up its history on their own". Whether or not Avital's solution can truly be called a solution is moot; nevertheless, it illustrates a general approach for combining history of mathematics and mathematics education, a strategy called elsewhere (Fried, 2001) the strategy of addition, namely, a strategy whereby history of mathematics is added to the curriculum by means of historical anecdotes, short biographies, isolated problems, and so on. Another general strategy which also answers the teachers' question "Where do I find the time to teach history?" is the strategy of accommodation (Fried, 2001), namely, using an historical development in one's explanation of a technique or idea or organizing subject matter according to an historical scheme. This strategy finds time by economizing, that is, teaching history and the obligatory classroom material at one stroke. Accordingly, Katz (1995), for example, suggests introducing the logarithm following Napier's geometric-kinematic scheme, arguing that it brings out functional properties of the logarithm important for precalculus students.

What is essential to observe is that however one solves the problem of time the obligatory classroom material must not be neglected. For this reason, in Avital's scheme, it is history of mathematics that must be pursued after school; history of mathematics is not "the topic you are teaching." For Katz, on the other hand, history can be taught as part of the lesson because it brings out ideas important for precalculus students—presumably, ideas belonging to the non-history-oriented curriculum. The point is that the pressure of a practical constraint, like time, forces one to subordinate history to the essential mathematics which teachers are committed to teach, that is, to algebra, geometry, calculus and the other subjects students need for more advanced study of mathematics as well as for engineering and the exact sciences. The mathematics educator must filter out from the history of mathematics what it is relevant from what is irrelevant and what is useful

[4] In this connection, the linguist Yishai Tobin refers to the Saussurean revolution that never took place! (Tobin, 1990, p.13).
[5] The ideas in this section are explored in much greater detail in Fried (2000, 2001).

from what is not. Indeed, a sign of this subordination is the more than occasional references to history of mathematics as something to be *used*,[6] rather than, I would add, as something to be studied in its own right.

The situation in which history is used to promote modern ends (in our case, the teaching of modern mathematics) is recognizable to the historian; historiographers call this way of doing history 'anachronical' (Kragh, 1987, p.89) or 'Whiggish' (Butterfield, (1931/1951). Herbert Butterfield, who invented the latter term,[7] says that "The study of the past with one eye, so to speak, upon the present is the source of all sins and sophistries in history, starting with the simplest of them, the anachronism...And it is the essence of what we mean by the word 'unhistorical'" (Butterfield, (1931/1951, pp.30-31). That statement may seem extreme, yet it is crucial to emphasize that the vigor of Butterfield's objection is not merely the vigor of a purist; it is a statement of what it means to do history at all. The commitment to avoid anachronism, to avoid measuring the past according to a modern scale of values, goes to the very heart of the historical enterprise. The history of ideas, which includes the history of mathematics, is dedicated to understanding how ideas change and, therefore, must begin by viewing past thought as truly different from modern thought. Writing about the difficulty of understanding Greek thought, W. C. Guthrie makes the nature of this task clear. He say, "…to get inside their minds requires a real effort, for it means unthinking much that has become part and parcel of our mental equipment so that we carry it about with us unquestioningly and for the most part unconsciously" (Guthrie, 1975, p.3).[8]

Using history of mathematics in mathematics education thus entails a clash of commitments. Mathematics educators are committed to presenting *modern* mathematics[9]; they are committed to students' understanding of the mathematical techniques and concepts used so powerfully in so many applications and fields of study; they are committed to students feeling at home with modern mathematics and recognizing it as a tool within their power to use. Historians of mathematics are interested in *shaking off* modern mathematics, "unthinking much that has become part and parcel of [their] mental equipment," as Guthrie put it; they are committed to seeing how the mathematics of the past diverges from mathematics as it is understood today; for them, mathematics is never just mathematics, a coherent body of knowledge that changes only by accumulation. If it were not for this clash of commitments, one might well argue that using the history of mathematics in the service of mathematics education is no different than, say, using examples from physics to illustrate mathematical concepts. As it is, it appears that the difference between history of mathematics and the ends of mathematics education reflects two opposing sets of norms—in fact, two different ways of looking at mathematics.

Thus, in Fried (2001), the theoretical problem was presented as a dilemma: if one is true to the commitments of mathematics education one is forced to adopt a Whiggist brand of history, *i.e.* history which is not history, whereas if one is true to the commitments of the history of mathematics one is forced to spend time on mathematical and philosophical ideas which may not be relevant to the general mathematics curriculum. While the argument may be sound, it is also deeply dissatisfying. One reason this is so is that it pits mathematics education against the history of mathematics—one discipline against another—instead of seeing two disciplines both focused on mathematics. Taking up this latter position one is led away from a conflict between disciplines and towards a view of mathematics itself as thing with two aspects: on the one hand, a

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[6] Indeed, one often sees the word "use" in the literature on the subject as in the titles "The Use of History of Mathematics In Teaching", "Using History in Mathematics Education", "Using the History of Calculus to Teach Calculus", "Using Problems from the History of Mathematics in Classroom Instruction", "Improved Teaching of the Calculus Through the Use of Historical Materials", to name a few.

[7] The term refers to a stream of British political history in which events of the past were conceived as steps progressing inexorably towards the democratic ideals that the Whigs held dear—as if the Whig party was the true *telos* of English history!

[8] This is also implicit in Collingwood's famous dictum that "the historian must re-enact the past in his own mind" (Collingwood, 1993, p.282ff).

[9] Counterexamples may exist, but judging from the great majority of mathematics curricula this does not seem to me a naïve generalization.

beautiful and coherent system of ideas, and, on the other hand, a living human creation whose ideas and ends are always changing and being redefined.

It is in clarifying this view of mathematics, as well as clarifying the basic dilemma, that Saussure's semiological ideas are helpful. For, to the extent that mathematics is a sign system, or at least exists within a sign system, Saussure would say that mathematics *must* have this double nature. Certainly, this will not solve all the difficulties in combining mathematics education and history of mathematics, but it points the way to a view of mathematics that makes an understanding of the history of mathematics as well as the usual school subjects *essential* to understanding mathematics as a whole. So, let us turn now to Saussure's theory of signs.

2. SAUSSURE'S SEMIOLOGY

The chief work by which we know Saussure's thought is the *Cours de Linguistique Générale* (Saussure, 1974).[10] The name of the work already hints at its revolutionary character. For it was not a work about a particular language, like Greek or Latin, nor some generalization from a particular language, like classical studies of grammar, nor yet a comparison between languages; for Saussure, linguistics meant *general* linguistics and as such it aimed to understand, in the words which end the book, "…language, considered in and for its own sake." Ironically, this end could not be attained without seeing language within an domain going beyond language, a science which "studies the life of signs at the heart of social life" (Saussure, 1974, p.33). This science of signs Saussure called *semiology*. More specifically, he writes:

> [Semiology] would teach us in what signs consist and what laws rule them. Since it does not yet exists, one cannot say what it will be; but it has the right to be, its place has been determined in advance. Linguistics is only a part of this general science; the laws that semiology will discover will be applicable in linguistics, and linguistics will thus find itself linked to a domain well defined within the ensemble of human realities (faits humains)" (*ibid*)

So, what is a sign? A linguistic sign is the result of coupling a concept with an 'acoustic image' (*ibid*. p.98). Schematically, this coupling can be represented by the following figure given in the *Cours*:

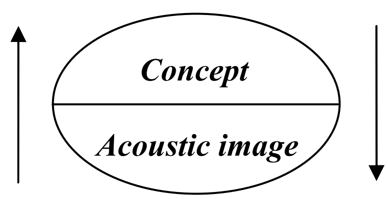

Fig.1

To understand the figure, it must be stressed, to start, that the 'acoustic image' is not itself a sound but a mental pattern of a sound.[11] Once formed, then, a sign exists as a 'psychological entity', with the 'acoustic image' no less in the mind than the concept. Moreover, as the arrows in Saussure's figure show, concepts and 'acoustic images' are mutually formative: concepts are not preexisting things *named* by means of sounds, and sound patterns *by themselves* are not embodiments of concepts; each justifies the other.[12] It may have been to highlight this fact that Saussure, almost immediately after introducing the notion of a sign, shifted his

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[10] The work was not written by Saussure himself but was a compilation of notes by two students, Charles Bally and Albert Sechehaye, who attended Saussure's lectures in general linguistics at the University of Geneva during the years 1906-1911. The book was published in 1916, three years after Saussure's death.

Unless noted otherwise, the translations from the *Cours* are my own.

[11] It must be understood, also, that while the 'acoustic image' may be a word, it does not have to be; it may be, for example, an intonation, or part of a word, or a phrase.

[12] It is Saussure's view that sound and thought are as inseparable in language as are the two sides of a single piece of paper, to use his metaphor (*Cours*, p.157).

terminology from concept and 'acoustic image' to 'signified' (the *signifiè*) and 'signifier' (the *signifiant*).[13] It is clear, though, that the generality of the latter terms also serves to bring out the generality of the sign and distinguish it from the specifically linguistic sign.

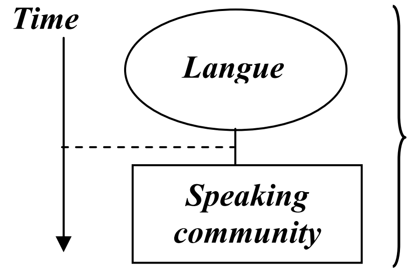
For Saussure, the crucial and most fundamental property of signs is that they are *arbitrary*, that is, no natural, logical, or any other universal law determines what signifier will be coupled with what thing signified. The reason this is so important for Saussure is that if it were not so one would have to admit that there is some kind of *natural* language; one could then identify that language as Greek or Latin or Sanskrit, and linguistics would again be the study of a particular language instead of language in general. The arbitrariness of signs, therefore, is not only the condition for a true diversity of languages, but also, paradoxically, for Saussure's dream of a general linguistics.
But this arbitrariness, as Saussure points out more than once, is not capriciousness; the sign is not subject to anybody's whim. Why is this? It is because of the way signs rest on a social foundation, or, in the case of linguistic signs, on a community of speakers. Thus Sausssure says, "Language at no moment, and contrary to appearances, exists except as a social fact, for it is a semiological phenomenon" (p.112). Indeed, because there is no determinative law of correspondence that joins signifier to signified the formation of the sign can only be a social act, just as the cementing of a tradition is. An individual can change a sign no more easily than an individual can alter a tradition; signs like traditions are made immutable by the "collective inertia" of society; they become, after they have been socially formed and internalized, social and psychological facts.

And yet societies change and languages change; signs too must change, for, like society itself, signs and the systems which comprise them exist in time. Time plays a double role: the semiological choice, the social coupling of signified and signifier, is fixed over time, but within time the relationship between signified and signifier shifts[14] and new signs are formed. Thus Saussure initially tempts the reader with a picture (illustrated by a diagram to that effect) of language as an entity resting on a social foundation, but he then shows that, without time, that picture is incomplete. He summarizes his position as follows:

> Since the linguistic sign is arbitrary, it may seem that language [here and throughout this passage the word is *la langue*], thus defined, would be a free system, organizible by will, dependant only on a rational principle. Its social character, considered in itself, is not directly opposed to this point of view. Without doubt, collective psychology does not operate on purely logical material; it must take into account everything which bends thinking (*fléchire la raison*) in practical relations between individuals. And yet it is not this that prevents us from viewing language as a simple convention, able to be modified to the liking of those interested; it is the action of time combined with that of social force; no conclusion is possible except that outside of the passage of time (*la durée*) linguistic reality is not complete.
>
> If one takes language in time, without the community of speakers (*la masse parlante*)—say, an isolated individual living over the course of many centuries—one could perhaps have no change; time would have no effect on it [language]. On the other hand, if one considered the community of speakers without time, one would not see the effect of social forces at work on language. In order to keep things close to reality we must add to our first scheme [captured by the diagram alluded to above] some sign indicating the passing of time:

[13] Initially, Saussure says only that the shift is to prevent confusion between the 'acoustic image', or whatever else calls the concept to mind, and the sign itself; however, subsequently he says that the new terms have the advantage of showing the difference between the parts of the sign and how they fit together into a unity. This supports my claim that in changing his terminology Saussure had something more fundamental in mind than convenience.

[14] Semiological change, for Saussure is always of this form, always a *shift* in the relationship between signified and signifier ("un déplacement du rapport entre le signifié et le significant" (p.109)).



(Fig. 2)

> So, language is not free because time allows social forces to exercise their effect on language, and one comes to the principle of continuity, which precludes freedom. But continuity necessarily implies change, a shift of relations of greater or lesser extent. (p.113)

The passage just quoted ends a chapter in the *Cours* entitled "Invariability and Variability of the Sign" ("Immutabilité et Mutabilité du Signe"). But how can a sign be *both* invariable and variable? It looks as if the notion of a sign leads us into a contradiction. To find our way out of the contradiction, Saussure's remark in the passage above that "if one considered the community of speakers without time, one would not see the effect of social forces at work on language" is helpful. An a-historical view of language, that is, a view of language at a single moment in the life of a society, presents language as a static perfectly coherent system of signs[15]; the effect of social forces is not evident because only the interrelations among signs, and not their formation, is apparent. When we allow history to enter the picture, the picture completely changes. Language appears from the historical point of view always in flux; we perceive that the "river of language flows without interruption" (p.193); it is in this view that we see society *making* its semiological choices, as opposed to the previous view in which we see the result of the choices having been made. So, it is not that language is characterized by contradictory properties but that it can be seen from two points of view. Saussure calls these, respectively, the *synchronic* and *diachronic* viewpoints.

Saussure pictures these viewpoints as two axes: the synchronic viewpoint is indicated by the axis of simultaneity (AB in fig. 3) and the diachronic view point by the axis of succession (CD in fig. 3).

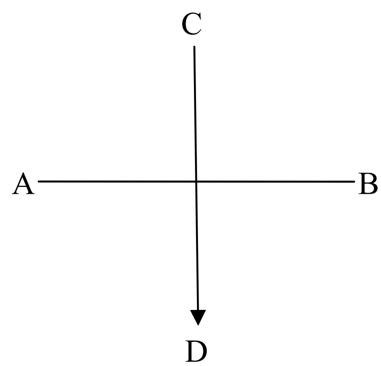


Fig. 3

Naturally, it is only the axis of succession which has an arrow attached to it; every line perpendicular to the axis of succession, that is, every line parallel to the axis of simultaneity, is a *static* linguistic state. The essential characteristic of this scheme is that, by having two perpendicular axes, it is made graphically clear that the

[15] By 'language' I am referring to what Saussure means by *la langue*, the 'language system', as opposed to the set of language acts, *la parole*.

viewpoints are irreducible.[16] But other aspects of the analogy are also suggestive. For example, note that different states, in this representation, are lines with no points in common. According to Saussure, a sign system changes sign by sign. Change, in other words, is never initiated from the system itself but from an individual sign. However, because the signs form a system, a change in a single sign must cause the entire system to change. He says, by comparison, that if the weight of a planet in the solar system were to change, the entire solar system would have make a corresponding adjustment (p.121). Saussure's favorite analogy in this regard, however, is that comparing the succession of synchronic states of a sign system with the succession of positions in a chess game (pp.125-127).[17] The transition in a chess game from position to position occurs as single moves of the chess pieces are made. However, with each move the entire position changes, that is, the relative value of each piece changes: what was safe is now in danger, what was insignificant now threatens to win the game, what was protected as essential is now sacrificed, and so on. So, the lines representing successive states of the sign system have no points in common because although most of the signs in different states may *look* the same, the change in state being initiated by changes only in individual signs, the relative values of the signs within the system have all shifted[18]; in this sense, *all* of the signs change with every change of the sign system.

The representation of the synchronic and diachronic viewpoints as pair of axes brings out, as I have said, the important recognition that the one viewpoint cannot be reduced to the other. It must also be said, however, that language, or any other sign system, cannot be understood entirely by means of one viewpoint or the other alone; the diachronic viewpoint, say, may take on a subsidiary role to the synchronic in one's study of language, but it can never be completely eliminated from one's considerations. Saussure makes this clear with yet another beautiful analogy. He says that the relationship between the two different viewpoints is like two different sections of a plant stem, one crosswise and one longitudinal.[19] Neither section could ever look like the other, yet they are interdependent. Moreover, neither section alone shows the plant in its entirety (nor could any other single section); it could never said that one section is the 'true' section, that is, that one section more truly represents the plant.

Synchrony and diachrony are essential semiological facts of life; without them one would be left with a contradictory notion of invariable signs that nevertheless evolve. But one must not get the impression that Saussure introduced synchrony and diachrony for the sake of logical consistency alone. On the contrary, that there be perfectly coherent semiotic states which are at the same time products of history is an unassailable requirement of any semiological system. For if anything at all is true about such systems it is that they are meant for communication and that they are produced by intelligent beings within a society, particularly, by human beings in human society; but where there is communication there must be coherence, and where there is human production there is history. With that, we must return now to our main concern, the problem of history of mathematics and mathematics education.

3. MATHEMATICS EDUCATION AND THE HISTORY OF MATHEMATICS IN THE LIGHT OF SAUSSURE'S NOTIONS OF SYNCHRONY AND DIACHRONY

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[16] By contrast, Jakobson, it should be pointed out, did not see the division between the two points of view as absolute, but always intermingling and interacting (see Holenstein, 1976, pp.25-33).

[17] Saussure points out one weakness in the analogy, namely, that the changes in a chess game, unlike those in language, follow the *intentions* of the players. Harris (1987) points out further weaknesses in the analogy, the most serious of which, in my opinion, is that the "possible states of the board and possible moves are alike governed by the rules of chess, which exist independently of the course of any particular game" (Harris, 1987, p.93). But this does not effect the main points of the analogy: 1) that the state of the board changes move by move and 2) that with every move the entire position changes.

[18] It ought to be mentioned that the word 'value' here is carefully chosen by Saussure. He calls language "a system of pure values" (p.116). One wants to replace the word 'value' with 'meaning', but 'meaning' hints at something that stands outside the system; 'value' is always something relative to other elements of the system; one might say the 'value' of a sign is its 'relative meaning'.

[19] Not only a description but also a drawing of the stem is included in the *Cours* (p.125).

By now the application of Saussure's ideas to the problem of mathematics education and history of mathematics ought to be fairly obvious. In teaching mathematics we present a synchronic view of the subject. We teach concepts, techniques, and procedures as parts of a fixed coherent system: when we teach the meanings of words such as 'function', 'derivative' or 'continuity' or use certain kinds of figures or representations in the classroom we are engaged in a semiological activity in which we have firmly in mind how these signs—for they *are* all signs—are to be placed in a system of mutual relationships. It is not at all surprising, therefore, that when speaking about mathematical understanding, Skemp (1978), for example, emphasized relational understanding and compared it to learning a map; I would stress too that the image implies that there is a genuine unchanging map to be learned.[20] There is nothing reproachable about this. The problem arises when we take up the history of mathematics, that is, a diachronic view of mathematics, without shifting ourselves away from the synchronic view which guided us previously. The result of this is that the history of mathematics becomes viewed not as an account of different ways of thinking, of truly distinct systems of thought, but as an account of a single expansive non-temporal system whose ideas are linked by immutable relations.

Saussure was well aware of the distortions that can arise from failing to distinguish adequately the synchronic from the diachronic viewpoints, but he was also aware of how easy it is to fall into the trap of *not* distinguishing between them. He writes: "Synchronic truth so agrees with diachronic truth that one may confound them or consider it superfluous to make a distinction between them" (p.136). To show what can go wrong, he takes as an example a common explanation for how the Latin verb, *faciō* ('I make') is related to the Latin verb *conficiō* ('I bring about, complete, make ready'). The relationship is explained by saying that the short *a* in *faciō* becomes *i* in *conficiō* since the position of the *a* in the compound *conficiō* is not in the first syllable. Thus, the relationship is understood as a direct one:

faciō ⟶ *conficiō*

But this is, indeed, a distortion of the relationship: the *a* in *faciō*, Saussure maintains, never *became* the *i* in *conficiō*. The way the situation must be understood is that there were "two historical periods and four terms" (pp.136-137): in one period, period A, the language community employed a pair of related signs, *faciō* and *confaciō*, while a second period, period B, it employed a second pair of related signs, *faciō* and *conficiō*. In the passage from period A to period B, the sign *faciō* remained the same while *confaciō* was transformed into *conficiō*. Saussure represents the situation schematically as follows:

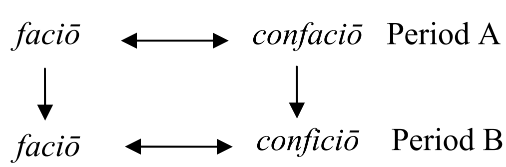

Fig. 4

Is it is important to add, however, that although *faciō* has remained the same in the passage from period A to period B it has a new identity[21] within the system of signs; for this reason, I believe, Saussure speaks of *four* terms and not three. The point is important for it reemphasizes Saussure's position that while the sign system changes sign by sign each such change brings with it a transformation of the entire system.

Mathematics is particularly prone to the confusion Saussure speaks of; it is not at all uncommon in mathematical contexts that "synchronic truth so agrees with diachronic truth" that one is tempted to account

[20] This comment could also be made with respect to Davis's analogy between understanding in mathematics and the piecing together of a puzzle (Davis, 1992)—one must assume that the puzzle is a fixed, though perhaps, ultimately unattainable whole. Incidentally, Toulmin (1960) similarly likened the building of a scientific theory to the making of a map. From which it follows that to the extent that we present students with scientific theories (and *a fortiori* mathematical theories) we are presenting them with a map to be grasped.

[21] Saussure would say it has a different 'value' within the system, as described above.

for mathematical change by quasi-logical explanations befitting terms in a single synchronic plane. To make the comparison with Saussure's discussion clearer, take the example of *faciō* and *conficiō* as a model and imagine there to be a synchronic plane representing the mathematics of some period in the past, historical period A, and another plane representing the mathematics of the present. In the first plane, two signs, *a* and *b*, exists within the mathematical community of period A (see fig. 5); the signs contrast with one another and, therefore, define one another. In the second plane, two signs, *a′* and *b′*, similarly exist within the present mathematical community. With *a* and *a′* as corresponding terms (*b* and *b′* may or may not correspond), like *faciō* in period A and *faciō* in period B above, a connection is made between *a* and *b′* by projecting these terms, as it were, into an ideal a-temporal plane, so that the sign *a* from the past appears to be related *directly* to the present sign *b′*:

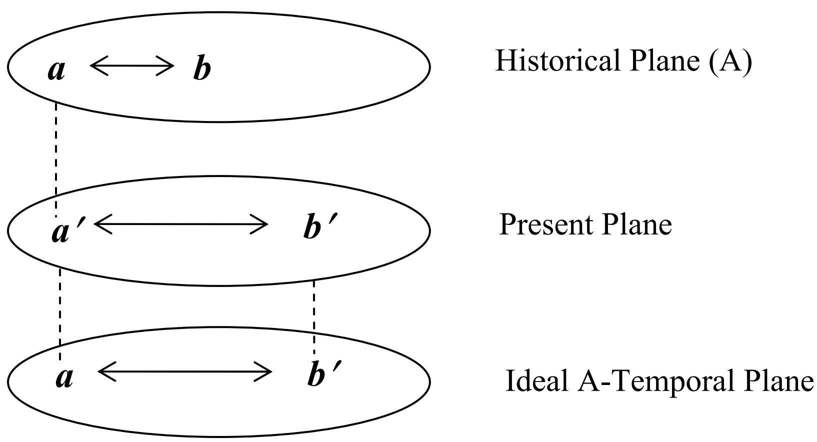

Fig. 5

Projecting the historical and present planes onto one ideal plane produces not one but two distortions. First, as in Saussure's discussion, it distorts the process by which the sign *b′* is born, that is, it produces a distortion from the point of view of diachrony. However, it also distorts the *synchronic* relationships in the historic plane, for it contrives a direct relationship between sign *a*, belonging to period A, and a sign b′, not belonging to period A. This is, in fact, none other than what was described above as 'anachronism'. For this reason, although we have referred to the diachronic viewpoint as the historical viewpoint, it is this second distortion which is the most serious one for the historian. Indeed, in trying to understand the processes by which mathematical ideas grow, the historian of mathematics must understand, first of all, how such ideas appeared to the mathematical communities using them; one might say the historian of mathematics, or any other historian of ideas, is interested in synchronies of the past.[22]

The problem is that to justify the introduction of history of mathematics into the classroom mathematics educators teaching a standard curriculum must be sure that the historical material is *relevant* to the curricular subjects. In effect, they are *forced* to relate *a* to *b′* and to ignore the relationship between *a* and *b*. Think of the difficulty of speaking about Euler's notion of a 'function' in a modern classroom studying functions.

[22] It should be kept in mind that I am using the words 'synchrony' and 'diachrony' as Saussure uses them. In historiography, by contrast, a 'diachronic' approach indicates something closer to what I have described here, namely, an approach directed towards "synchronies of the past." Thus, Kragh (1987, p.90) writes, "…in the diachronical perspective one imagines oneself to be an observer *in* the past, not just *of* the past. This fictitious journey backwards in time has the result that the memory of the historian-observer is cleansed of all knowledge that comes from later periods."

The modern notion of a function entails any pairing between all the elements of one set and the elements of another with the sole condition that any given element of the first set is paired with only one element of the second. To illustrate the generality of this formulation, it is common for teachers and textbooks to present functions having graphs not only like fig.6a but also like fig. 6b and 6c:

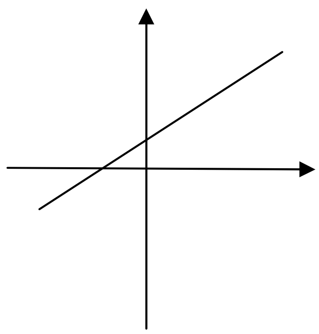

Fig.6a

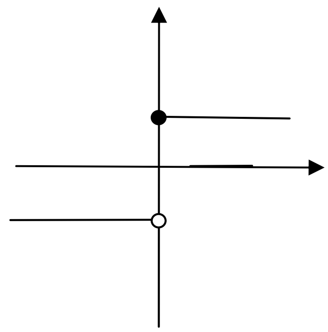

Fig.6b

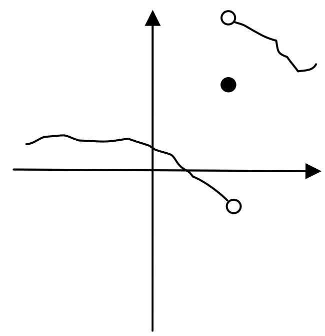

Fig.6c

To bring out this point further, the teacher might want to mention that a function, as Euler viewed understood it, had to be given by a single analytic rule,[23] so that, for him, fig. 6b and fig. 6c would not be considered true functions; in fact, functions defined on a split-domain would not have been considered true functions, nor, in general, would discontinuous functions have been (see Sui, 1995; Kleiner, 1993; Youschkevitch, 1976). The pedagogical strategy here is obvious: students have to expand their view of a function to include functions such as those represented by fig. 6b and 6c; this process is eased and made tangible by showing that the history of mathematics itself followed the same course having similarly to expand its view of what could be called a function. There is some truth in saying that the history of mathematics followed this course and some sense in the general pedagogical strategy. But this apparent correspondence between what happens in history and what steps students must follow to learn some mathematical concept in fact only demonstrates Saussure's observation that "Synchronic truth agrees to such a degree with diachronic truth that one may confound them or consider it superfluous to make a distinction between them."

The problem is, again, one of forgetting that our 'function', 'continuity', 'discontinuity' and Euler's 'function', 'continuity', 'discontinuity' are *different* signs in separate synchronous sign systems. Consider the statement "Euler's notion of function was such that discontinuous functions, in general, were not considered by him to be true functions." An 18th century mathematician would agree with us that this is a true statement. But what does it mean? What *we* mean by it is that functions like those in fig. 6b and 6c were, in general, excluded by Euler. For Euler, however, 'function' signifies a single analytic expression, a single rule, involving a variable and numbers; 'continuous' signifies the continuous application of the rule to the variable throughout the domain of the variable. Thus, the relationship between 'function' and 'continuous' is not one of genus and species as it is for us but almost one of definition. For the same reason, the 'discontinuity' of the function represented by fig. 6b has not to do with the break in the graph but with the break and change in the rule at $x=0$. Hence, for Euler, the function $f(x)=1/x$, for example, is 'continuous' (Kleiner, 1993). The trouble with functions on a split-domain, on the other hand, is that they are 'discontinuous'.[24] Moreover, it

[23] Euler's 1748 definition of a function in his *Introductio in analysin infinitorum* was: "Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitibus constantibus" ("A function of a variable quantity is an analytic expression composed in any way from this variable quantity and numbers or constant quantities")

[24] More precisely, Euler called such functions 'mixed'; a 'discontinuous' function was function that was not governed by a rule, and, for this reason, problematic as a 'function' at all.

cannot be said strictly that the notion of function *expanded* to include discontinuous functions; it ought to be said, rather, that 'continuity' has shifted its meaning.[25]

It if all this were only a matter of historical accuracy or completeness, one could suggest simply that teachers 'learn the facts' or, since there is no end to accuracy, that they 'just do their best' and not worry too much about satisfying everyone's standards of accuracy. But the difficulty lies deeper than this. Whether or not one recalls Euler's notion of a 'mixed function' *is* a matter of historical completeness; however, the failure to recognize 'function', 'continuity', and so on as signs in a synchronous sign system different from ours is in some sense to miss the entire historical picture. But this, in its turn, means not understanding present day mathematics as itself set within a specific synchronous sign system existing at a certain moment in time. Why this is important is connected with what it means for there to be such sign systems to begin with: to reiterate, it means that there is a coherent and functioning system of communication produced and used by a community of human beings. It is because of this that studying history of mathematics, in the sense of studying past synchronies (and, accordingly, modern ones as well), can humanize mathematics while providing insight into mathematics as a system of ideas. And this being one of the central arguments adduced for incorporating history of mathematics in mathematics education, one begins to see what is at stake in grasping fully the diachrony/synchrony distinction.

4. IMPLICATIONS FOR TEACHING: THE CHALLENGE OF A HUMANISTICALLY ORIENTED MATHEMATICS EDUCATION

Despite the discussion above, one might still ask whether Saussure's semiological ideas have really placed us in a better position to confront the problem of history of mathematics and mathematics education, or have they only suggested a restatement of the problem? Have we escaped the dilemma between adopting a synchronic view of mathematics where we ignore or, worse, distort history and adopting a diachronic view where we risk leading students away from the main tasks and goals of the standard curriculum? Has not turning to Saussure made the dilemma even more trenchant? After all, it is Saussure who says that "The opposition between the two points of view—the synchronic and the diachronic—is absolute and brooks no compromise" (p.119) and that in studying language one must make a choice between the synchronic and diachronic approaches. Must a mathematics teacher, then, choose between 'mathematics' and 'history of mathematics'? To conclude that from Saussure would be to misunderstand his intention.

Saussure, as discussed in part 2 of this paper, took great pains to show that neither the diachronic nor synchronic view is alone the true view of language, that they are complementary views of language. When Saussure said that one must make a choice between the synchronic and diachronic approaches he only meant that when linguists take up linguistic questions they cannot disregard the point of view from which they frame and investigate them. I do not believe he thought linguists must commit themselves entirely to one or the other approach. But what about *teachers* of linguistics? For this, we have Saussure himself as an example, for we must not forget that the *Cours* records the work of a teacher. In view of this, it is obvious that Saussure believed that understanding 'language' entails understanding both the diachronic and the synchronic aspects of language. So, if the application of Saussure's ideas to the teaching of mathematics is truly valid, we must conclude that teaching 'mathematics' also demands presenting both its diachronic and synchronic aspects; far

[25] Cauchy, for example, pointed out in 1844 that 'mixed' functions such as

$$y = \begin{cases} = x, & x \geq 0 \\ = -x, & x < 0 \end{cases}$$

could be written as a single analytic expression, in this case $y = \sqrt{x^2}$. Hence, as Youschkevitch (1976, pp.72) put it, "the discrimination between *mixed* and *continuous* functions proved theoretically untenable" (Youschkevitch, 1976, pp.72). Therefore, attaching to' function' the idea of a 'continuous' rule became itself problematic.

from having to choose between 'mathematics' and 'history of mathematics' the teacher must give attention to both. This, then, is the first lesson we learn from Saussure.

The second lesson, which is deeply entwined with the first, concerns how teachers ought to treat mathematics as a subject. Recall, Saussure's semiological approach to language provided two important insights: 1) To the extent that the sign is arbitrary, the semiological process is cultural or social; 2) To the extent that, synchronically, signs form a system, semiotic systems have coherence and logic. Applied to mathematics, the second insight offers no great surprise; the first, however, tells us that mathematics, as a product of human activity, is a humanistic subject. Furthermore, Saussure makes us realize that these two insights are linked, for it is within the social framework that signs become fixed, that they become 'immutable'. The application of Saussure's thought to mathematics education suggests, then, not only that mathematics *is* a humanistic subject, but also that it *ought* to be viewed that way.[26] This, then, is the second lesson we learn from Saussure.

Of course, viewing mathematics education in a humanistic vein should not necessarily help students solve any specific mathematical problem. It should, however, inculcate in them a sense that words like 'creativity', 'inventiveness', 'vision' are appropriate for thinking about mathematics. It should show students that mathematics is something that human beings *do*—and this doing includes more than solving problems or proving theorems; it is also includes creating concepts and formulating ideas, that is, as human acts. This was not the approach taken in the example given above concerning functions and continuity. True, an historical figure, Euler, was introduced into the lesson; however, he was not portrayed as formulating a concept which he called a 'function', but only as taking a step towards the modern notion of 'function', as if the latter were somehow a 'natural' concept.[27] We should rather ask, what is the system of ideas that forces Euler to use the sign 'function' as he does? By doing so, we encourage students to ask what is the system of ideas that makes our use of the sign 'function' connected with the sort of representations in figs. 6a-c. In this way, we do not present a 'function' as some object given in advance, but as a sign produced by human beings and possessing a meaning to be interpreted.

The move to a humanistically oriented mathematics education is one answer to the problem of history of mathematics and mathematics education. Is it a radical answer? To answer this question recall that the dilemma faced by teachers of mathematics with regards to the history of mathematics took the form of a clash of commitments. If the move to a humanistically oriented mathematics education requires teachers' changing their commitments as mathematics teachers or altering the framework of standard curricula which express those commitments, then the answer is radical. But if we consider what has just been said about how historical ideas might have been brought into a discussion of functions without history becoming a-historical, we realize that history can play a part in the classroom without the material and focus of our mathematics teaching becoming radically altered. What is altered is a kind of background sense of the mathematical subjects we are teaching; the human origin of mathematical ideas, which the serious study of history brings out supremely, becomes *subsidiary*, to use Polanyi's term (Polanyi, 1958), to the study of the usual topics. Thus, a humanistic mathematics education will not deprive students of the knowledge of the 'state of the art' but will make them realize that the art is, indeed, in a certain, though not necessarily permanent, state.

[26] Somewhat different arguments for a mathematics as a humanistic subject are given in Fried (2003), as well as in the many papers found in White (1993).

[27] To say that there are 'natural' concepts (or, as Saussure says, 'panchronic' concepts), *i.e.* concepts which are not the result of human making, is to adopt a version of Platonism. As a philosophy of mathematics, Platonism is legitimate of course; however, since it assumes that mathematical concepts exist *independently* of human activity, Platonism cannot be consistent with the history of mathematics *qua* history.

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