Inverses – why we teach and why we need talk more about it more often!

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Abstract: This article examines the key role that the notion of inverses plays in numerous mathematical concepts.

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Introduction

The point of "inverse" gets down to "the inverse of which operation?". Different operations lead different answers about the inverse. The main reason why students are confused or make mistakes are that they are not clear which operation teachers refer to. In this writing, I first discuss the inverse of three types of operations: addition (its inverse is known as "opposite"), multiplication (its inverse is known as "reciprocal") and composition of maps or functions (its inverse is known as "inverse"). Then I introduce the concept of group, ring, and the field in order to challenge students to utilize their cognitive understanding of inverses in solving equations.

It was not until I had begun to teach trigonometry that I realized the students' understanding of inverse is not so concrete. Given $\sin 30° = \frac{1}{2}$, most students knew $\csc 30° = 2$; some shouted "you flip it! That's how you got it." It appeared that they understood the reciprocal of $\sin\theta$ is $\csc\theta$. A few days later, when asked the same group about the value of $\sec 60°$, a number of students responded $\sec 60° = \cos\left(\frac{1}{60}°\right)$. Intrigued by this, I did a survey of asking a sophomore class about the inverse of 2. Most students answered $\frac{1}{2}$. When asked the same question, a number of senior students, however, responded, "where is x and y?"

How many inverses are there?

When it comes to inverses, there seems to be a sequential understanding of inverse as students progress from pre-algebra to algebra and pre-calculus. Why are they confused and

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inadequate in doing inverse questions? The unfamiliar notation itself is a reason and they have not learned inverses in a more systematic way where each inverse is presented under different operation.

In the beginning, they learn the opposite of 3 is -3, the reciprocal of 3 is $\frac{1}{3}$, then ultimately the inverse of $f(x) = 2x + 1$ is $f^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$. Still, a math teacher encounters a student who mistake $f^{-1}(x)$ for $\frac{1}{f(x)}$ on a daily basis. The trouble continues to be epidemic in trigonometry where many students thinks that $\sin^{-1}\theta = \frac{1}{\sin\theta}$. The following was one of my pop quiz questions.

[Example 1]

6. Find a correct statement.

(A) $\tan^{-1} x = \cot x$ (B) $\sin\frac{1}{15} = \frac{1}{\sin 15}$ (C) $5\sin\theta = \sin 5\theta$

(D) $\sin\frac{2}{17} = \csc\frac{17}{2}$ (E) $\cos\frac{1}{15} = \frac{1}{\sec\left(\frac{1}{15}\right)}$

I was surprised only one third of the class in fact identified the answer choice (E) correctly. One may dismiss this problem as unfamiliarity with notations. In fact, if a teacher focuses on using correct notations, students began to do well with questions like example 1. Also, teachers are encouraged to discusses that the meaning of $f^{-1}(x)$ is the composition inverse of $f(x)$ as a function, not the reciprocal inverse of $f(x)$. Additionally, a number of students also find it more effective and consistent for a teacher to use notations such as $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$, instead of $\sin^{-1}\theta$, $\cos^{-1}\theta$ and $\tan^{-1}\theta$.

The followings (example 2; example 3) are is my lecture notes I used particularly with introduction of inverse functions because I thought that there should be an emphasis on a more concrete understanding of inverses under different operations.

[Example 2]

Quantity	**Operation**	**Inverse**
2	*Addition*	-2 (opposite)
2	*Multiplication*	½ (reciprocal)
y = 2x + 1	*Composition*	½ x – ½ (inverse)

[Example 3]

The inverse of $\tan\theta = \arctan\theta$.

(The inverse of $\tan\frac{1}{23} = \arctan\left(\frac{1}{23}\right)$)

The reciprocal of $\tan\theta = \dfrac{1}{\tan\theta} = \cot\theta$

(The multiplicative inverse of $\tan\theta = \dfrac{1}{\tan\theta} = \cot\theta$)

The opposite of $\tan\theta = -\tan\theta$

(The additive inverse of $\tan\theta = -\tan\theta$)

After emphasizing this and repeating the notations during a number of class activities, then any math teacher would realize students begin to pick up a correct notation and can develop the idea that inverse is a more comprehensive and a practical concept. Furthermore, the following example [example 4] is also a powerful case in point where students appreciate their understanding of inverse.

[Example 4]

$\sin x = 0.3$

$\sin^{-1}(\sin x) = \sin^{-1}(0.3)$

$x \approx 17.5°$

In fact, there is more to think about in example 4. In the beginning, teaching to a test, I experienced a huge success when I take an easy way out by just telling the students to use a short-cut key in a graphing calculator whenever they needed to find a measure of the angle. There is no doubt that most students easily solved a question like example 4. However, I soon found that the students still struggle with grasping the concept of inverse in the context of composite functions. Not surprisingly, they had a difficulty in finding an answer to the following question [Example 5].

[Example 5]

Find x and y.

$\cos^{-1}(3x) = 20°$

$\cot(x^2 + 1) = 3$

Although it appeared to be time-consuming to walk through the process with students by discussing the inverse functions, it turned out to be more effective in the long term. The following [Example 6] is an example of a discussion modeling how to solve questions like the second question in example 5.

[Example 6]

Question: Solve for x. Given $\cot(x^2 + 1) = 3$

Teacher: *It is an equation with a variable x. How would you isolate x?*

Student: *Take arc co-tangent of both sides. That gives back* $(x^2 + 1)$ *with the variable x.*

Teacher: *If you're not familiar with arc co-tangent, how would you change the equation.*

Student: *Rewrite the equation* $\tan(x^2 + 1) = \dfrac{1}{3}$ *because the reciprocal of* $\cot\theta$ *is* $\tan\theta$. *Then, take the inverse of tangent of both sides so that it returns* $x^2 + 1$, *which equals* $\tan^{-1}\left(\dfrac{1}{3}\right)$.

Teacher: *Try to show give the exact value of x.*

Student: $\left(x^2+1\right)=\tan^{-1}\left(\frac{1}{3}\right)$. *Then,* $x=\pm\sqrt{\tan^{-1}\left(\frac{1}{3}\right)-1}$

At the end of the day, I concluded that the students need to be presented with a variety of contexts where inverses plays important roles and that high school math teachers might need to spend more time discussing inverses and their roles in both arithmetic and advanced mathematics.

For example, in solving a simple linear equation, students need to be reminded of the fact that they use the concept of inverse everyday.

[Example 7]

4x – 5 = x + 7	
3x – 5 = 7	**cancel out x; additive inverse of x is -x**
3x = 12	**cancel out 5; additive inverse of -5 is +5**
x = 4	**cancel out 3; multiplicative inverse of 3 is 1/3**

In example 7, often we see a student saying "move -5 to the other side and in fact I used to explaining that way because that's how my math teacher taught me in Korea. But that explanation doesn't reveal much about the idea of inverse. In stead of instructing the student to switch the sign if a number needs to move the other side of the equation, a teacher should specifically mention that to cancel out a number, they need to the reciprocal or the opposite or the inverse function depending on the operations. That is, though informally, students need to understand that inverse is the quantity which cancels out a given quantity and that as a result, the operation gives the identity. Additionally, there are different kinds of inverses for different operations (i.e., addition, multiplication, composite, matrix multiplication, etc.)

Furthermore, a math teacher should be willing to re-teach the following [Example 8; Example 9] if necessary.

[Example 8]

Given 5

5 + 0 = 5 ; 0 + 5 = 5

Then, 0 is the identity under the additive operation because the identity returns the value.

Therefore, to formulate this idea, a + a^{-1} (=inverse of a) = 0

a^{-1} = –a (opposite or additive inverse of A)

It is also important to note that a^{-1} is being exclusively used for the multiplicative inverse.

[Example 9]

Given 5

$5\times1=5$; $1\times5=5$

Then, 1 is the identity under the multiplicative operation because the identity returns the value.

Therefore, to formulate this idea, $a\times a^{-1}=1$.

a^{-1} (=inverse of a) = $\frac{1}{a}$ (reciprocal or multiplicative inverse of A)

Sooner than later, all the efforts and hard work to introduce inverses, a group of math enthusiastic students may emerge with more fundamental questions. "What would happen if we get rid of inverses in math?" "How do inverses help to do mathematics? A succinct answer could be:

"To solve an equation." Ideally, it is the best time to introduce a basic group theory in terms of inverses.

The following is a collection of informal definitions of key terminologies in group, ring and field. (Woldfram MathWorld Websites)

A group G is a finite or infinite set of elements together with a binary operation (called the group operation) that satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. There must be an inverse of each element in the group.

A ring is a set S with binary operators $+$ and $\times$, satisfying the six conditions: Additive associativity, Additive commutativity, Additive identity, Additive inverse, Left and right distributivity, and Multiplicative associativity. Note that it is not necessarily that there is an inverse of reciprocal of nonzero element in the ring. Example of rings are integers without multiplicative inverse and even-valued integers without multiplicative identity.

A field is a special ring with additional three conditions: Multiplicative commutativity, Multiplicative identity, and Multiplicative inverse. Most notable is that there must be a multiplicative inverse for each non-zero element. Examples of fields is the complex numbers, rational numbers, and real numbers. (Dummit 2003)

The following is an example of an activity where students create a system of numbers such as a group. In the process, the students see the significance of inverse in constructing a number system.

[Assignment 1]

Create a binary operation using addition and multiplication in $\Re$.

(a) Define the binary.

(b) Determine whether there is an identity.

(c) If there is an identity, examine whether it is a group under this binary operation (i.e. whether the binary operation satisfies closure, associativity, identity and inverses.)

[Student Work Sample 1]

Define $A\#B = 3(A+B) - A \cdot B$ where $A, B \in \Re$

$A\#e = 3(A+e) - A \cdot e = A$

Then $e = \dfrac{2A}{(A-3)}$

<u>Teacher's comments</u>: How about 3? You must disallow 3, but then it contradicts your set, which is $\Re$. Also, e is an identity which must be independent of any elements. Here, $e = \dfrac{2A}{(A-3)}$ depends on A.

[Student Work Sample 2]

Define $A\#B = 7(A-B)$ where $A, B \in \Re$

$A\#e = 7(A-e) = A$

Then $e = \dfrac{6A}{7}$; $A^{-1} = \dfrac{43}{49}A$

To solve, $e\#A = 7(e-A) = A$

Then $e = \dfrac{8A}{7}$; $A^{-1} = \dfrac{41}{49}A$

<u>Student comments</u>: This group doesn't have the identity because A#e = e#A.

Teacher comments: e can't be the identity because it is dependent of elements. Also show that A # B = B # A

[Student Work Sample 3]

$A \Diamond B = A + B - A \cdot B$ where $A, B \in \Re$

$A \Diamond e = A + e - A \cdot e = A$

Then, $e = \dfrac{0}{1 - A} = 0$, $A \neq 1$

$e \Diamond A = e + A - e \cdot A = A$

Then, $e = \dfrac{0}{1 - A} = 0$, $A \neq 1$

This group is closed; if $A \in \Re$, $B \in \Re$, then $(A + B - A \cdot B) \in \Re$.
The identity is 0.
The inverse is $\dfrac{A}{A - 1}$ where $A \neq 1$
Associaitivity stands ; $(A \Diamond B) \Diamond C = A \Diamond (B \Diamond C) = A + B + C - AB - BC - CA + ABC$

Teacher's comments: Note that $A \neq 1$. Since A = 1 does not have inverse, $A \Diamond B = A + B - A \cdot B$ is not a group.

It is important to discuss why it is important to note that $A \neq 1$. A teacher needs to show (See Teacher Demonstration 1) what would happen if they allow A=1.

[Teacher Demonstration 1]

Solve it.
$1 \Diamond x = 7$
Then, 1 + x – x = 7
1 = 7, which is not possible.

Solve it.
$2 \Diamond x = 7$
2 + x – 2x = 7
-x = 5
x = -5

The students find it more interesting to see that using 1 in an equation gives an empty set as a solution. This demonstration is more valuable than dismiss the number 1 simply because $\dfrac{A}{A - 1} = \phi$ if A = 1. Then it would be interesting to see some students modify their group like the following:

[Student Work Sample 3 *Modified*]

$A \Diamond B = A + B - A \cdot B$ where $A, B \in \Re - \{1\}$

$e = 0$; $A^{-1} = \frac{A}{A-1}$

Note that if A and B are not 1, then A + B – AB is not 1 either.

[Teacher Demonstration 2]

Solve it. $5\lozenge x = 8$

(Solution A)

$5^{-1}\lozenge 5\lozenge x = 5^{-1}\lozenge 8$; $5^{-1} = \frac{5}{5-1} = \frac{5}{4}$

$x = \frac{5}{4}\lozenge 8$

$x = \frac{5}{4} + 8 - \frac{5}{4}\cdot 8 = \frac{-3}{4}$

(Solution B)

5 + x – 5x = 8

-4x = 3

x = $-\frac{3}{4}$

The Solution B is quick and more concise. The students solve it by following the definition of the operation, which is an important habit to keep in learning mathematics. The Solution A is, however, more intriguing to the students because they witness how the inverse plays out in solving an equation. Then it is also important to ask students in the Solution A whether it would be OK to do $5^{-1}\lozenge 5\lozenge x = 8\lozenge 5^{-1}$ instead of $5^{-1}\lozenge 5\lozenge x = 5^{-1}\lozenge 8$.

[Student Work Sample 4]

$5^{-1}\lozenge 5\lozenge x = 8\lozenge 5^{-1}$

$x = 8\lozenge\frac{5}{4} = 8 + \frac{5}{4} - 8\cdot\frac{5}{4} = -\frac{3}{4}$

Students then understand that why it is important to have the property of commutative. It is also valuable to mention that the number system we are using is a group of operations such as addition and multiplication that satisfies some basic rules (i.e., axioms). And it is a good idea to pose them a question such as what would happen if a number system doesn't obey commutative rules (see Student Work Sample 5]. Additionally, a teacher needs to note that if no identity is found, then consequently no inverse would exist either. (Ash 2006) If no inverse would exist, then there would be no way to solve an equation. (Mirman 1997) For example, for a new operation @ defined as A@B = 7(A – B). Then one figures out that there is no identity. Hence, no inverse. Then given 8@x = 2, no one would be able to solve this equation using the operation, @ because the inverse of 8 doesn't exist.

[Student Work Sample 5]

8@x = 2

7(8 – x) = 2

$$x = \frac{52}{7}$$

To solve, x@8 = 2

7(x – 8) = 2

$$x = \frac{58}{7}$$

To conclude, 8@x is not equivalent to x@8.

For more motivated students, the modular arithmetic is a good topic to be introduced after a math teacher helped the students build strong foundation in the previous topics. Zn is the algebraic structure of the set { 0, 1, 2, 3, ..n-1 } with the operations of addition and multiplication. For example, our clock system: Z_{12} has elements {0, 1, 2, 3, 4, ….11} which is equivalent to {12 o'clock, 1 o'clock, ….11 o'clock}. The following table gives the set of possible elements for each operation.

[Table 1; addition]

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

[Table 2; multiplication]

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	**1**	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	**1**	6	11	4	9	2	7
6	0	6	0	6	0	6	0	6	0	6	0	6
7	0	7	2	9	4	11	6	**1**	8	3	10	5
8	0	8	4	0	8	4	0	8	4	0	8	4
9	0	9	6	3	0	9	6	3	0	9	6	3
10	0	10	8	6	4	2	0	10	8	6	4	2
11	0	11	10	9	8	7	6	5	4	3	2	**1**

Well prepared by the previous topics, most students can find that Z_{12} an additive group with the identity, 0 and the inverses after they verified the axioms. Likewise, the students would try to verify Z_{12} is a multiplicative group. But soon they found out that only the elements 1, 5, 7 and 11 have their inverses.

[Student Work Sample 6]
Question (A): Is Z_{12} a group?
Solution: Z_{12} an additive group, but Z_{12} is not a multiplicative group.

Question (B): Solve x + 8 = 1 ($x \in Z_{12}$) using the table 1 of addition.
Solution: $8^{-1} = 4$
x = 1 + 4
x = 5

Question (C): Solve 5x = 4.
Solution: $5^{-1} = 5$
x = 5*4
x = 8

The Student Work Sample 7 shows the students' investigation on Z_5.

[Student Work Sample 7]

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Closure OK
Associativity OK
Identity = 0
Inverses OK

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Closure OK
Associativity OK
Identity = 1
Inverses OK!
(Note: 0^{-1} doesn't exist, but the requirement is every non-zero element has an inverse.)

Summary: Z_5 is an additive group; Z_5 is a multiplicative group; Z_5 is a ring; Z_5 is also a field.

The following is student generated question and solutions written by them.

[Student Work Sample 8]

Question (A)
Solve it. $4x \equiv 2 \pmod 5$
$4^{-1} \equiv 4$
$4^{-1}\, 4\, x \equiv 4^{-1}\, 2$
$x \equiv 3 \pmod 5$

Quesition (B)
Solve it. $3\ (4x + 1) \equiv 2 \pmod 5$
$3^{-1}\ 3(4x + 1) \equiv 3^{-1}\ 2$
$4x + 1 \equiv 4$
$4x + 1 +$ (add.inverse of 1) $\equiv 4 +$ (add.inverse of 1)
$4x \equiv 3$
$4^{-1}\ 4\ x \equiv 4^{-1}\ 3$
$x \equiv 2 \pmod 5$

Conclusion
These additional activities would enhance the students understanding of inverses and appreciation of its crucial role in establishing a number system. If a math teacher only teaches that the opposite of 4 is -4, the reciprocal of 3 is $\frac{1}{3}$, and the inverse of $f(x) = x + 1$ is $f^{-1}(x) = x - 1$, undoubtedly a student would struggle with conceptualizing inverses in more advanced mathematics such as modular functions. There is no doubt that the students would get lost and have weak understanding because there are so many notions of inverses. This in fact calls for a need for a theory that unifies all these notions. It is one of the reasons why people want to introduce the concept of group, ring and the field. The abstract math concepts are very powerful, it could be applied to different concrete cases although it pays the price of being “abstract.” It is the beauty of mathematics.

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