Learning Mathematics with Understanding: A Critical Consideration of the Learning Principle in the *Principles and Standards for School Mathematics*[1]

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Abstract: *Learning with understanding has increasingly received attention from educators and psychologists, and has progressively been elevated to one of the most important goals for all students in all subjects. However, the realization of this goal has been problematic, especially in the domain of mathematics. To this might have contributed the fact that, although the vision of students learning mathematics with understanding has often appeared in curriculum frameworks, this vision has tended to be poorly described, thereby offering limited support to curriculum development and policy. The Learning Principle in the Principles and Standards for School Mathematics, an influential mathematics curriculum framework in the United States, seems to make an effort to break this tradition by offering a research-based description of what is involved for students to learn mathematics with understanding. In this article, we examine the extent to which the Learning Principle meets this goal in light of seminal scholarly work on learning mathematics with understanding. By solidifying some key ideas set forth in the Learning Principle and by identifying ideas for further consideration, the article contributes to the development of better descriptions in curriculum frameworks of issues related to promoting meaningful learning in school.*

1. Introduction

> How is it that there are so many minds that are incapable of understanding mathematics? Is there not something paradoxical in this? Here is a science which appeals only to the fundamental principles of logic, to the principle of contradiction, for instance, to what forms, so to speak, the skeleton of our understanding, to what we could not be deprived of without ceasing to think, and yet there are people who find it obscure, and actually they are the majority. (Poincaré, 1914, pp. 117-118)

Henri Poincaré's statement captures eloquently both the inextricable relation between mathematics and understanding, and the difficulty that learning mathematics with understanding

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entails. While learning mathematics with understanding has increasingly received attention from mathematics educators and psychologists and has progressively been elevated to one of the most important goals of the mathematical education of all students, the realization of this goal has long been problematic. Many factors might account for this, such as teachers' knowledge and pedagogy, the curriculum, etc. In this article, we consider one of those factors, namely, the curriculum, focusing on one curriculum framework's description of issues related to promoting meaningful learning in school. This focus is important because, although the vision of students learning mathematics with understanding has often appeared in curriculum frameworks, this vision has tended to be poorly described, thereby offering limited support to curriculum developers.

The *Principles and Standards for School Mathematics*, a mathematics curriculum framework recently released by the National Council of Teachers of Mathematics (NCTM, 2000) in the United States (US), seems to make an effort to break this tradition. The *Standards* document offers, in a section called the Learning Principle (NCTM, 2000, pp. 20-21), a research-based description of what is involved for students to learn mathematics with understanding. This article has been motivated by the increased value assigned nowadays to learning mathematics with understanding as a principal instructional goal for all students and by the high potential of the *Standards* to influence curriculum development.

Our primary goal in this article is to consider critically the research-based vision about meaningful learning in school mathematics that is elaborated in the Learning Principle (LP). We pursue this goal by discussing both strengths and weaknesses of the LP in light of scholarly work that could be considered seminal with regard to the theme of learning mathematics with understanding. Although our article is about a US mathematics curriculum framework, the discussion we conduct can be of interest to a broader audience. There are two primary reasons for this. First, the case of the US can be seen as indicative of the current trend in many countries to emphasize meaningful learning in school curricula across all subject areas (especially in mathematics and science). Second, the *Standards* have influenced the authors of curriculum frameworks in many countries.

The article is structured into two sections. In the first section, we provide evidence in favor of some key ideas advanced by the LP. Our discussion 'unpacks' these ideas, elaborating how they find support from existing research (part of which is not referenced in the LP). In the second section, we discuss some points that, although important and warranted by research, are insufficiently addressed in the LP. The issues raised in this section represent recommendations for how the LP could be enhanced. By solidifying some key ideas set forth in the LP and by identifying ideas for further consideration, the article contributes to the development of better descriptions in curriculum frameworks of issues related to promoting meaningful learning in school. A word of caution here is that our examination of the LP focuses on what is said or being referenced *in* the LP. One could argue that our analysis should have considered the entire *Standards* document in which the LP is embedded. We decided not to do that because one of the most interesting aspects of the LP is its seeming effort to describe – in a short and self-contained text – the essence of the *Standards'* vision with regard to students' learning mathematics with understanding.

The article as a whole can also be viewed as a survey of literature on learning mathematics with understanding, offering an interpretation and synthesis of some important points on this topic where there is consensus. Our focus on issues of consensus is deliberate, as we believe that a curriculum framework should primarily be judged based on its potential to communicate to curriculum developers well-established points that can serve as guiding principles in their efforts to design effective curricula.

2. Evidence in Favor of Key Ideas Set Forth by the Learning Principle

The LP supports the claim that learning with understanding is both *essential* and *possible* in school mathematics. The argument in favor of meaningful learning in school mathematics was made and supported experimentally as early as the 1930s (Brownell, 1935, 1940, 1947), and has been elaborated since then by many proponents of learning with understanding (e.g., Skemp, 1976). It has also been corroborated by the results of many recent studies of varying instructional and theoretical approaches. These studies: (1) collectively emphasize the importance of having meaning related to learning activities of students of varied ages, backgrounds, and abilities (Cobb et al., 1991; Fennema & Romberg, 1999; Hiebert & Wearne, 1993; Silver & Stein, 1996; Zohar & Dori, 2003), and (2) reveal the need for more instructional attention to sense-making as part of school mathematics instruction (e.g., Schoenfeld, 1988; Silver et al., 1993). In support of the LP, this mounting body of research suggests that *all* students can understand and apply important mathematical concepts. Also, this scholarly work emphasizes the merits of students developing conceptual understanding, and stresses the importance of the powerful connections established between procedures and concepts when one practices this kind of learning.

An important point set forth by the LP is that memorization of facts or procedures without understanding often results in fragile learning. This remark corresponds to research which has shown that mastery of facts and rote performance of procedures are not sufficient in thinking mathematically (Schoenfeld, 1988), getting the right answers does not necessarily imply mathematical proficiency (Erlwanger, 1973), and learning computational formulas is a poor substitute for developing understanding of the underlying concepts (Pollatsek et al., 1981). What is perhaps more important is that the LP goes a step further to note that conceptual understanding is only one out of at least three major components of proficiency, the other two being factual knowledge and procedural facility, and that the alliance of the three makes them usable in powerful ways. This claim finds support from research that has demonstrated the compatibility and close interrelation between factual and procedural competence, and learning with understanding (Bransford et al., 2000; Hiebert & Carpenter, 1992; Silver, 1987). Silver (1987), for example, emphasizes that pure forms of either conceptual or procedural knowledge are seldom exhibited, if ever, and that "it is the *relationship* between the knowledge types that gives one's knowledge the power of application in a wide variety of settings" (p. 183; emphasis added).

Related to the above is the LP's emphasis on the relationship between students developing mathematical understanding on the one hand and making connections among mathematical ideas and procedures on the other. Hiebert and Carpenter's (1992) definition of mathematical understanding in terms of the way knowledge is structured illuminates this relationship:

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p. 67)

Well-connected and conceptually grounded ideas enable their holders to both remember them and see them as part of a larger whole within which each part shares reciprocal relationships with other parts (Resnick & Ford, 1981; Romberg & Kaput, 1999; Schoenfeld, 1988, 1992). In addition, ideas with these characteristics are fluently accessed for use in new situations (Skemp, 1976) and empower their holders with the ability of *transfer* – that is, the ability to use what they have learned in new and unfamiliar problems, and to learn related information more quickly (Bransford et al., 2000; Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Resnick & Ford, 1981; Schoenfeld, 1988). In sum, learning mathematics with understanding involves *making connections* among ideas; these connections are considered to facilitate the transfer of prior knowledge to novel situations. Transfer is essential because most new problems require solution via previously learned strategies; it would be impossible for one to become mathematically competent if each problem required a separate strategy.

Hiebert and Carpenter (1992) emphasize that, "[o]ne observation that assumes near axiomatic status in cognitive science is that students' prior knowledge influences what they learn and how they perform" (p. 80). The LP makes a strong point about the power of using children's experience and prior knowledge in learning mathematics with understanding. Research suggests that students bring to school a considerable amount of knowledge and experience, and that students construct meaning for a new idea by relating it to ideas that they already know or have experience with (Bransford et al. 2000; Gagnon & Collay, 2001). In the particular domain of mathematics, research shows that children begin to construct mathematical relations long before they come to school. These early forms of knowledge can serve as the basis for developing several components of the formal elementary mathematics curriculum and for further expanding children's understanding of mathematics (Carpenter et al., 1981, 1996; Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Schoenfeld, 1992). For example, the work of Carpenter and his colleagues demonstrates that, by the time children begin school, they already have a considerable knowledge store relevant to arithmetic. Although children may lack the symbolic representations of addition and subtraction, they have experiences of adding and subtracting numbers of items in their everyday play, and they can solve a wide range of addition and subtraction problems. If children's knowledge is tapped and built on as teachers attempt to teach them the formal operations of addition and subtraction, it is likely that children will acquire a coherent and thorough understanding of these processes. Mack's (1990) study with eight sixth-graders offers a different example. It shows that students at this level of schooling can construct meaningful algorithms in learning fractions, given that instruction builds appropriately upon their informal/prior knowledge on this domain.One other important point made by the LP is that learning mathematics with understanding can be promoted through students' engagement in *problem solving* activities. Several researchers emphasize that curriculum activities that engage students in problem solving reflect an emphasis on learning mathematics with understanding (Fennema et al., 1999; Romberg & Kaput, 1999; Schoenfeld, 1992). For example, Fennema et al. (1999) note:

> Because the goal of mathematics education should be the development of understanding by all students, the majority of the curriculum should be composed of tasks that provide students with problem *situations*. Two reasons support this claim. The first is that mathematics that is worth learning is most closely represented in problem solving tasks. The second is that students are more apt to engage in the mental activities required to develop understanding when they are confronted with mathematics embedded in problem situations. (p. 187)

From an epistemological point of view, problems are the source of the meaning of mathematical knowledge. As Vergnaud (1982) remarks, "[n]ot only in its practical aspects, but also in its theoretical aspects, knowledge emerges from problems to be solved and situations to be mastered" (p. 31). Intellectual productions become knowledge only if they prove to be efficient and reliable in solving problems that have been identified as being important practically (they need to be solved frequently) or theoretically (their solution allows a new understanding of the related conceptual domain). Inextricably related to engaging in problem solving is getting involved in activities related to *mathematical reasoning* and *proof*: exploring patterns; making, testing, and evaluating conjectures; and developing mathematically sound arguments for or against mathematical statements. Several scholars have elaborated on the connection between learning mathematics with understanding and reasoning and proof. Ball and Bass (2003) emphasize that "mathematical reasoning is inseparable from knowing mathematics with understanding" (p. 42). In the same spirit, Hanna and Jahnke (1996) note that "[p]roof in its full range of manifestations is … an essential tool for promoting mathematical understanding in the classroom" (p. 877).

A final point we wish to highlight is that the LP considers both the *cognitive* and *social* aspects of learning. The consideration of both psychological and sociological conceptions of learning agrees with the current trend of integrating these two perspectives (see, e.g., Cobb & Bauersfeld, 1995; Yackel & Cobb, 1996). With regard to the psychological approach to learning, the LP acknowledges the constructivist idea that understanding is a continuing activity of individuals organizing their own knowledge structures, a dynamic process rather than an acquisition of categories of knowing (Confrey, 1994; Gagnon & Collay, 2001; Piaget, 1948/1973; Pirie & Kieren, 1994). The LP also notes that learning with understanding supports the creation of autonomous learners – that is, learners who "can take control of their learning by defining their goals and monitoring their progress" (NCTM, 2000, p. 21). The notion of autonomy goes back at least to the work of Piaget (1948/1973) who proposed that the main goal of education should be the cultivation of learners' autonomy. With regard to the sociological perspective to learning, the LP (NCTM, 2000) advances the idea that "[l]earning with understanding can be further enhanced by classroom interactions, as students propose mathematical ideas and conjectures, [and] learn to evaluate their own thinking and that of others," and notes that "[c]lassroom discourse and social interaction can be used to promote the recognition and connection among ideas and the reorganization of knowledge" (p. 21). Examples of this kind of classroom environments can be found in the several research reports (e.g., Ball & Bass, 2003; Lampert, 1986, 1990; Yackel & Cobb, 1996).

3. Recommendations for Enhancing the Learning Principle

As the discussion in the previous section suggests, the LP summarizes well some key ideas regarding learning with understanding in the context of school mathematics. However, there are some other important ideas that, although warranted by research, are insufficiently addressed in the LP. In this section, we discuss four such ideas and provide recommendations about how these issues could be addressed in future versions of this or other curriculum frameworks.

Our first point is that the LP does not discuss the important idea that what we consider a desirable learning outcome (e.g., development of understanding or acquisition of procedural fluency) determines the relative worth of instructional methods. The way in which Brownell (1935, 1947) and Skemp (1976) – two proponents of teaching and learning for understanding – approach this issue can serve as a useful model for the LP.

Although both scholars favor meaningful learning and teaching of arithmetic and believe that thorough understanding of computational procedures cannot be achieved without a solid conceptual basis, neither of them rejects non-meaningful ways of teaching and learning arithmetic. Brownell (1935) notes that "*drill* is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse" (p. 19; emphasis added). Skemp (1976) mentions three advantages of teaching *instrumental mathematics*, that is, rules without reasons: (1) Within its own context, instrumental mathematics is usually easier to understand; (2) The rewards are more immediate and apparent; and (3) Because less knowledge is involved, instrumental thinking can often help one achieve the right answer more quickly and reliably than relational thinking. Furthermore, both Brownell and Skemp advance the argument that, depending on what learning outcomes are valued, different methods should be employed; accordingly, there is no absolute instructional method. Brownell supports this argument by using his experimental findings regarding the learning of the subtraction algorithm for 'borrowing' from tens. On the one hand, he found that the method of 'decomposition' was more effective than 'equal additions' when the desired learning outcomes were the development of students' understanding and the enhancement of their ability to transfer their knowledge. On the other hand, he found evidence that the equal additions method was superior to decomposition when both were taught mechanically. Skemp (1976) expresses a similar idea when he notes, for example, that, "[i]f students are being taught instrumentally, then a 'traditional' syllabus will probably benefit them more" (p. 156). The foregoing remarks indicate that the two researchers recognize that a fundamental question in teaching is what are the desired learning outcomes. Given the interdependence between desirable learning outcomes and appropriate instructional methods, teaching for understanding might not always be the most appropriate instructional method.

The results of Hiebert and Wearne's (1993) investigation about the relationships between teaching and learning place value and multi-digit addition and subtraction in six second-grade classrooms support a similar idea to the one advanced in the previous paragraph. Hiebert and Wearne found that students in classrooms that emphasized the construction of relationships between place value and computation strategies received fewer problems than their more traditionally taught peers, spent more time with each problem, were asked more questions to describe and explain alternative strategies, and showed higher levels of performance by the end of the year on most types of activities. These findings initially appeared to be at odds with some conclusions derived by earlier work (e.g., Good at al., 1978; Leinhardt, 1986). This is because earlier work suggested that teachers who stimulate high rates of student achievement, with regard to basic skills at the primary grades, have the following characteristics: they teach quickly paced lessons, they ask more recall than process/explanation questions, and they present more problems per lesson than do novices. This 'discrepancy' could be attributed to the relationship between desirable learning outcomes and instructional procedures. The characteristics of 'effective teachers' derived from studies prior to Hiebert and Wearne (1993) were probably limited to traditionally taught classrooms (see Brophy & Good, 1986). As Hiebert and Wearne (1993) note, "these characteristics may relate to higher achievement if compared with other classrooms using a *similar* (but not as effectively implemented) instructional approach" (p. 422; emphasis added).

We turn now to our second point, which relates to the issue of knowledge transfer from one situation to another. While the LP captures well the idea that conceptually grounded knowledge is more likely to be transferred to new problem situations, it does not consider situations where transfer does *not* happen. One way to address this issue would be to draw on the extensive body of research that considers the situated character of learning (Boaler, 1998; Carraher et al., 1985, 1987; Lave et al., 1984). The theory of *situated cognition* explains why transfer does not happen in terms of the idea that learning is linked to the situation or context in which it takes place. This theory accounts, for example, for cases where adults do not use their school-learned arithmetic in grocery shopping; adults often do not recognize mathematically similar situations, thus choosing procedures depending more on the context rather than on the mathematical aspects of the tasks (Lave et al., 1984). Situated cognition also accounts for cases where children are more successful in solving arithmetic problems in word context than when solving equivalent but purely symbolic problems (Carraher et al., 1987), or for ones where children demonstrate superior performance when solving problems in the market as compared to the school-like setting (Carraher et al., 1985).

Related to the above is our third point. The LP does not discuss the possibility of students' prior knowledge and experience becoming a burden in their future learning of mathematics, thereby appearing to suggest that prior knowledge and experience always facilitate subsequent learning. However, as Bransford et al. (2000) point out, "[p]revious knowledge can *help* or *hinder* the understanding of new information" (p. 78; emphasis added). One way to account for this issue would be for the LP to caution the readers that prior experience, even when correct in the context it was generated, might not necessarily be readily applicable in new contexts. Several studies help exemplify this point. Bell et al. (1981) showed that children have difficulties with verbal problems about decimal numbers because of beliefs they acquired from their previous engagement in other mathematical domains and that are resistant to change. Two such beliefs, probably originating from students' experience with whole numbers, are 'multiplication makes bigger' and 'the smaller number must always be divided into the larger.' Fischbein et al. (1985) propose that there are certain types of intuitive models linked with arithmetic operations and used in the initial instruction that "become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behaviour even after the learner has acquired formal mathematical notions that are solid and correct" (p. 16; in original the whole segment is emphasized). The repeated addition and partitive models are two such examples for multiplication and division, respectively. Instructors often choose these models "as initial didactical devices because they correspond best to the mental requirements of elementary school children at the concrete operational period and because they provide the most natural way of understanding the new concept" (Fischbein et al., 1985, p. 15). While these models are not wrong, they are incomplete and do not capture all the different meanings of multiplication and division. Along similar lines, Resnick et al. (1989), based on an analysis of children's errors as they learn decimals, conclude that "errors are a natural concomitant of students' attempts to integrate new material that they are taught with already established knowledge" (p. 8).

We get now to our last point. Although the LP considers mathematical learning both at the level of the individual and the social group, it seems to overlook the important role of the *cultural* context in which learning takes place. According to learning theories in the domain of cultural discourses, "learning and knowing, whether focusing on the level of the individual or the social group, can only be understood when considered in the broader cultural context" (Davis et al., 2000, p. 69). Jerome Bruner (1996), a leader in the field of *cultural psychology*, argues that one "cannot understand mental activity unless [one] takes into account the cultural setting and its resources, the very things that give mind its shape and scope" (pp. x-xi). The crucial role of culture in shaping an individual's understanding suggests that learning can take different forms for students with different cultural backgrounds. In turn, this emphasizes the importance of 'culturally relevant pedagogy' (Ladson-Billings, 1994) to cultivate learning with understanding in classroom environments with diverse student populations.

4. Conclusion

While learning mathematics with understanding is an important instructional goal for all students, forms of classroom mathematics practice that foster meaningful learning seem to deviate from the norm, at least in US mathematics instruction (Hiebert at al., 2003; Manaster, 1998). This state of affairs is in part due to the challenges that arise from trying to make learning with understanding a consistent part of all students' everyday mathematical experiences.

One promising way to gain leverage on helping students learn mathematics with understanding is to equip teachers with curriculum materials (student textbooks and teacher editions) that provide them with the necessary guidance. This argument finds support in the large body of research that suggests that the mathematical activity that takes place in classrooms, including teachers' decisions about what mathematics tasks to implement and how, are mediated through the curriculum materials they use (Beaton et al., 1996; Burstein, 1993; Nathan et al., 2002; Porter, 1989; Remillard, 2000; Romberg, 1992; Schmidt et al., 1997; Stein et al., 1996; Zaslavsky, 2005). But the design of curriculum materials that can be used by teachers to engage their students in meaningful learning is a complex endeavor and so the guidance that curriculum frameworks (such as the NCTM *Standards*) can offer to curriculum developers on this issue is crucial.

Some questions that curriculum frameworks need to address in regard to integrating understanding into a coherent conception of mathematics learning in school curriculum materials are the following: What might be the relationship among factual knowledge, procedural facility, and understanding in mathematical learning? What might be the role of problem solving and reasoning and proof in learning mathematics with understanding? What might be the influence of students' prior knowledge and experiences in learning with understanding, and how might these be addressed or used effectively by instruction? What might account for knowledge transfer in some situations and what might inhibit this process in others? What might be the role of the broader cultural context in which students' learning unfolds, and how might this relate to individual and social factors? What might be the relationship between learning outcomes and instructional methods, and what might this relationship imply for learning mathematics with understanding?

The LP makes a serious attempt to position itself on many of the above issues, thereby guiding in significant ways the efforts of curriculum developers who are committed to improving the quality of students' learning of mathematics. By recognizing both the importance and the complexity of the goal to describe issues of promoting meaningful learning in school, we used seminal scholarly work in this area to examine the extent to which the LP meets this goal. Our examination revealed that the LP substantially captures some key points elaborated in the literature related to learning with understanding but insufficiently addresses some other important points. Our discussion in this article of the points that are insufficiently addressed in the LP is not meant to devalue the importance and potential contributions of the LP to curriculum development; rather it is meant to suggest ways in which the LP could be further improved.

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