The Nature of Proof in Today's Classroom
Erica Lane

Harold Fawcett. ***The Nature of Proof***. New York City: Bureau of Publications, Columbia University, 1938. Re-printed by the National Council of Teachers of Mathematics in 1995.

Throughout the world, children are taught new ideas and concepts in a variety of ways, and the advocates of the different approaches claim that their method is the most effective and profitable for the students. In addition to the different teaching styles, each individual learns the material in a distinct manner. Teachers, therefore, must be aware of the diverse techniques of both teaching and learning in an effort to optimize the learning experience for the greatest number of students. This endeavor is even more difficult than it sounds. Imagine a classroom with twenty-five students. Some students learn better in a lecture setting while others learn best through visuals. Still others learn by working in pairs or groups, and some students need hands-on lessons to encourage the thought process. Numerous other learning styles exist, including the child who just does not want to learn. Everyday, teachers across the globe are faced with these situations, and researchers have attempted to find a teaching style and classroom atmosphere that is conducive to all students' needs and abilities.

An idealist would ask: Is there one approach to teaching that will cover all of the learning styles and engage the most (if not all of the) students? If there is, does this style work for all subject areas? Unfortunately, there is not one set method in which to teach all subjects nor is there one best method of teaching a single subject. If one existed, all teachers would be using it and there would be no need for further research. So, researchers today continue to conduct experiments that explore different teaching techniques. However we have a classic example of an effective teaching technique in mathematics education literature, which is both progressive and innovative by modern standards, namely Fawcett's (1938) two-year study regarding the teaching of geometry. In this book review, I will investigate Fawcett's research, citing specific examples and methods of teaching and offer new ways to apply his theories in today's classroom. In doing so, I hope that teachers will re-examine this mathematics education classic and consider its applicability to today’s classroom.

In high school geometry classes in the U.S, students are presented geometry in generally one of two styles. Traditionally, geometry has been a class separate from algebra and offered in between and Algebra I and Algebra II courses. Some schools still present mathematics in this sequence, but many schools have turned to “integrated” mathematics in which geometry, algebra, and trigonometry are combined into a series of four books. Recent examples of integrated curricula are the NSF-funded SIMMS, Core-Plus etc. Students typically take Integrated I – IV and possibly Calculus. In these classes, students are encouraged to see mathematics as a whole and use algebra and geometry to solve problems. Students take these classes in sequence, unless they need a slower curriculum or and accelerated program, at which time other texts are available.

Although these are the methods employed in the United States, both have apparent shortcomings that are causing the students to lose information that could be otherwise gained. For example, in the separated classes, students have little opportunity to connect geometry to algebra and algebra to geometry. To students, these areas are separate entities, and bridging the gap between the two subjects seems unrealistic. Even though these are specialized topics in mathematics, the two are connected, and students should explore more links between the two subjects. Recently, mathematics teachers and researchers felt the same way, so the implementation of the integrated books relieved the unconnected-ness of algebra and geometry, supposedly. The original NSF funded, research based and standards aligned integrated curricula fulfilled the promise of an authentic integration of the various strands of mathematics. However the "mainstream" K-12 textbook industry was quick to jump on the new mantra of integrated mathematics and to produce curricular materials, which tries to appease the traditionalists as well as the reformists. This hodge-podge "integrated" curriculum is very different from the NSF-funded curricular material (such as SIMMS and Core-Plus) and is now found in numerous school districts across the nation, especially those in which the administrators strive for innovation but face resistance from parents and "didactically" traditional teachers. The unfortunate consequence of this attempt to appease everybody is units and sections in the integrated books choppy and disconnected, far-removed from the vision of creating this new curriculum. The books do contain algebra, geometry and some trigonometry, but there is little relationship between the different sections, which generates confusion for the students. Without connecting one idea to the next, mathematics can be a very difficult field to learn and master. In general, these are the different ways that schools in the United States teach high school students geometry. The question then is do other techniques exist that may prove to be more beneficial in the teaching of geometry and proofs to students?

In 1932, Harold P. Fawcett, designed an experiment to test the usefulness of geometric and proof practices and how extensive work in geometry can lead to proficient and successful "transfer" of ideas and thought processes as they relate to the world outside of the classroom. The word "transfer" here is used very differently from the behaviorist conception of this construct. During Fawcett's study, only one method of teaching geometry existed in the United States, for the integrated boom had not yet hit schools. For his experiment, Fawcett established two groups, the experimental group and the traditional group. Placement of students did not depend on mathematical ability, standardized test scores, age, or gender. Students from grades nine through eleven were randomly placed in one of the classes and then asked their opinions regarding mathematics and the class that they were about to take. Each class met four times a week for forty minutes each day.

In the traditional classroom, the teacher taught the students demonstrative geometry using the standard method of the 1930s. The students learned from a geometry textbook that gave them definitions, axioms, and Euclid's theorems (in today's terms), and then they proved other theorems using the givens. Step by step, students would prove new theorems, and then use these to prove future theorems. Strictly guided by the text, students had little freedom to explore geometry on their own, and the curriculum had a rigorous structure, unlike the experimental group. This geometry was presented as a separate subject from algebra (which was the trend for decades after the study, as well), just as in the experimental group.

In the experimental group, students had the vast world of demonstrative geometry to explore at their own pace and with their own ideas. Conventional textbooks, much to the surprise of the students, became obsolete, and the students used their abilities and experiences over the two-year period to create their own textbooks. With nothing more than a little direction from the teacher, students worked individually, in pairs, and as a class to define terms, create assumptions, and prove theorems. The class became almost like a democracy (pun-intended for today's times!). All, or the majority of students, had to agree with the newly defined terms (or terms deemed "undefined") and proven theorems before they could be added to the textbooks of each student. Fawcett (1938) described the "principles and methods" that the teacher of the experimental group followed as:

1. No formal text is used. Each pupil writes his own text as the work develops and is able to express his own individuality in organization, in arrangement, in clarity or presentation and in the kind and number of implications established.
2. The statement of what is to be proved is not given the pupil. Certain properties of a figure are assumed and the pupil is given an opportunity to discover the implications of these assumed properties.
3. No generalized statement is made before the pupil has had an opportunity to think about the particular properties assumed. This generalization is made by the pupil after he (sic) has discovered it.
4. Through the assumptions made the attention of all pupils is directed toward the discovery of a few theorems, which seem important to the teacher.
5. Assumptions leading to theorems that are relatively unimportant are suggested in mimeographed material, which is available to all pupils but not required of any.
6. The major emphasis is not on the statement proved, but rather on the method of proof.
7. The extent to which pupils profit from the guidance of the teacher varies with the pupil and the supervised study periods are particularly helpful in making it possible to care for these variations. In addition individual conferences are planned when advisable. (p.62)

Using these teaching techniques, the teacher (who happened to be Fawcett, himself) acted more as a moderator than an instructor and allowed the students to teach themselves and each other throughout the two years.

After two years, both classes took tests on the fundamentals of geometry and their ability to analyze non-mathematical. The students were evaluated on six criteria: "record of scores made by pupils on the Ohio Every Pupil Test in plane geometry; results of paper-and-pencil tests on the nature of proof applied to non-mathematics situations; contributions of students illustrating situations to which habits of thought developed in their study of the nature of proof had transferred; parents' observations concerning improvement in the critical thinking of their child; record of six observations made by college seniors; and students' observations concerning improvement in their ability to think critically" (p.102).

On the *Ohio Every Pupil Test* as well as the nature of proof test, the experimental group scored well above the average of all students who took the tests and improved greatly from the first time they took the nature of proof test. The experimental group showed a significant increase in their ability to “transfer” critical thinking skills to other environments, and the majority of parents agreed that the logical and critical thinking skills of their children had been enhanced through this class. Finally, the reports from the college seniors as well as the pupils themselves represent a positive, intellectually stimulating experience from the two years in the experimental, demonstrative geometry course. These students were placed in a situation that varied from the usual classroom setting and rose to the occasion by expanding their abilities and skills and applying these to their lives after the class ended. The students in this class, like the mathematician Euclid did two thousand years ago, increased their own knowledge while opening doors for their peers at the same time. Their individual textbooks may never be published, but their experience has been regarded as a most beneficial way for students learn demonstrative geometry.

Since Fawcett's approach to teaching students geometry and methods of proof demonstrated a great method via which students became involved with the material, it is important to explore specific aspects of his techniques and attempt to find ways in which this can be used in today's classrooms. It seems that if this method is as effective as Fawcett claims it is, teachers should try to incorporate it into their classrooms. However, since Fawcett published the results of his study 1938, little change is evident in the teaching of geometry until the integrated units in the 1990s. One may wonder why his methods did not catch on in the United States and if it is even possible to teach with such ambiguity in the structure of geometry curriculum.

To begin his class, Fawcett and his students discussed definitions and the need for defining terms so that each student, as well as the reader, has a firm basis and understanding of the material in the new textbooks. Before jumping into words such as point, line, circle, etc., students were asked to define "school," "outstanding achievement," "foul ball," and "tardy." Although these are words that individuals use everyday without specifically defining, the students had a difficult time agreeing on these and other definitions (p.31-34). It is important to note that the class spent four weeks establishing the fact the need for precise definitions on which all students agree.

After struggling with defining these terms, Fawcett allowed his students to brainstorm and determine how they wanted to approach defining geometric terms and the study of geometry. The class continued in this way for weeks as the students created lists of undefined terms and defined terms. These ideas followed this structured path (p.42):

1. Undefined Terms
    a. "…selected and accepted by the pupils as clear and unambiguous."
    b. "No attempt was made to reduce the number of undefined terms to a minimum."
2. Definitions
    a. "The need for each definition was recognized by the pupils through discussion…"
    b. "Definitions were made by the pupils."
3. Assumptions
    a. "Propositions which seemed obvious…were accepted as assumptions…"
    b. "These assumptions were made explicit by the pupils…"
    c. "No attempt was made to reduce the number of assumptions to a minimum."
    d. "The detection of implicit or tacit assumptions was encouraged…"
    e. "The pupils recognized that…the formal list of assumptions is incomplete."

Many would agree that this is a great way to get students involved in their own education, an idea that was missing in classrooms in the 1930s and is still missing in many classrooms today. However, as stated above, the students took four weeks just to establish the relevance of definitions in the world around them as well as in the world of mathematics. With the new stipulations on student learning and success, this seems like an unrealistic amount of time to apply to ideas that are non-mathematical. Nevertheless, I believe that teachers could use such an approach in teaching geometry today.

In a graduate level course in which I recently enrolled, ten students worked to define the first terms in Book I of Euclid's *Elements*. This endeavor proved to be a challenge for my class, and we labeled many terms such as point, line, and ray undefined, just as the students in Fawcett's study. The task was time consuming, but we decided that much of our struggle came from the fact that we had all been introduced to the geometric terms, assumptions, and theorems in previous classes influencing the creation of our geometry. I would argue that students in a beginning geometry class in high school will not have the same bias, and therefore will be able to reach conclusions regarding some terms and definitions. The lists of definitions and assumptions to memorize and the perfectly proved, textbook theorems have not yet spoiled their minds. These students have the ability to start from the ground and create a solid structure of geometry or other mathematical emphases with guidance from a teacher who has a firm grasp on the subject.

It is not reasonable to expect a teacher to cover an entire geometry curriculum in this way while conforming to the recently passed *No Child Left Behind Act*, but I encourage a trial period in which students in geometry (or algebra, trigonometry, calculus or even in other disciplines) are allowed to explore the specifics of the mathematics with little direction from the teacher and more influence from their peers and their previous experiences. Instead of spending a month defining terms to use for the entire two-year curriculum, I have created a lesson plan that incorporates just slight direction from the teacher and depends highly on the participation of students and their interactions with one another. This plan, or other related plans, are recommendations and may not be productive in every class.

The teacher should act as a moderator. The students should be in control of the lesson. Make sure books are closed; this is entirely from their previous knowledge.

1. *Introducing Angles*
    a. *Ask students how they define an angle. Do all the students agree? Put all of the suggestions on the board. Now, ask the students what words on the board should be defined. For example, with the definition of an angle, words like "point," "ray," and "distance" probably fall into the definition.*
    b. *Break students into groups of three or four and have them create the "ideal" definition of an angle while defining all of the ambiguous terms in their definition. Moderate the time based on students' involvement in the assignment.*
    c. *Each group will present their definitions to the class, and the other students will have a chance to agree or disagree with the group's rationale.*
    d. *Put the words acute, obtuse, straight, vertical, complementary, and supplementary on the board. As a class, have the students brainstorm the definitions to these different angles. Find common ground among the students to establish one list of definitions (and pictures) in which most (all) students agree.*
    e. *The most important part of this lesson (I think), is asking the students why this is important and relevant and how they think it is going to be used in mathematics. When and where are angles used in mathematics? Outside of mathematics? What professions use angles in their everyday routine? How have angles been used in historic mathematics? Have students write a paragraph or two using the defined terms and applying angles to the world.*

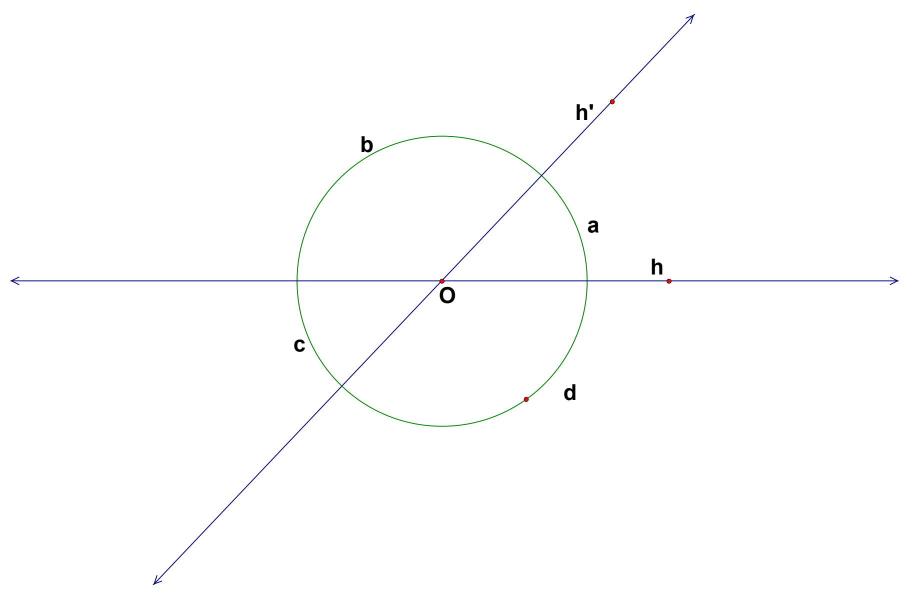
In Fawcett's study, the students used their experience of defining terms and applied their new knowledge to situations that were non-mathematical. How, for example, can the word restaurant be defined? It was important for the state of Ohio, at one point, to use the word restaurant in a law, but what is a restaurant? The state said that a restaurant is "a place of business where 50 per cent or more of the gross sales accrue from the sale of food-stuffs consumed on the premises" (p.46). Students were then asked to consider five questions, two of which are presented below (p.47):

1. The gross sales of one of the White Castles in Columbus is approximately $12,000 a year, all of this coming from the sale of food. Some of this food is eaten where purchased while the remainder is eaten elsewhere. If the amount eaten elsewhere is $6158.40, is the White Castle rightly called a restaurant?
2. How would you decide whether or not a combined ice cream parlor and soda fountain which also serves light lunches is a restaurant?

The reader should bear in mind that Fawcett's study took place over a two-year period, so lengthy discussions such as the previous example were possible. Rarely in mathematics class is there time for discussing the definition of a restaurant. However, defining terms in laws may prove to be beneficial to students in history, political science, or English classes. It is important to note that although this study was done with a geometry class, Fawcett's techniques can be modified to fit a variety of disciplines. Fawcett used these examples to emphasize the importance and defining terms so that all readers agree with the definitions. His students established an appreciation for definitions and assumptions, which enhanced their participation in the geometry portion of the class.

Finally, Fawcett's students jumped into the realm of geometry and were presented with scenarios often in the form of sketches in which they had to draw conclusions and create a list of assumptions to add to their texts. The book has a wealth of examples, I will present only one. At this point in the curriculum, teacher ability to lead the students without making conclusions for them becomes crucial. Fawcett had to introduce topics to his class in a fashion that encouraged a thoughtful and profitable discussion and led the students to discover something new.

For example, students were asked to state the properties of this figure when "h is in a fixed position and h' revolves about O in a counterclockwise direction." (p.52)



Students proposed the following properties (and others):
h and h' can be extended indefinitely.
Vertical angles are always equal.
There is a time when the four angles formed are equal.
Angle *a* is never smaller than 0°.

Some students agreed with these statements and others, and some students would not support these claims without further investigation and possibly proof. Fawcett continued in this manner for weeks until students had made significant additions to the assumptions portion of their texts. Then, students were introduced to the method of induction, which promotes forward thinking in mathematics and all disciplines. Students used induction to reason geometric proofs and to argue about non-mathematical statements including the Declaration of Independence and the assumptions used in this work. Working through these non-mathematical situations as well as geometric scenarios, students completed their textbooks in the two-year class and were tested, as described previously in the paper, against the traditional class.

In the end, students in the experimental group labeled twenty-three terms undefined, defined ninety-six terms, made fifty-five assumptions, and proved twenty-one theorems and an additional thirty-three, although not all students agreed with these others. Individually, students were rewarded by having a constructed a geometry text that was their own creation! After 182 hours (since little if no outside work was necessary), the students had achieved phenomenal goals and scored much higher on the tests than the group which was taught in the traditional manner.

Fawcett provided his students with an amazing technique to learn geometry and method of proof, but can his structure (which was funded and not under the scrutiny of *No Child Left Behind*) be used in today's classroom settings? Can teachers allow students to create their own mathematics without the aid of textbook and still meet the standards set forth by the school districts, state government, and federal government? Although recreating this setting is not impossible, it would be difficult. Seventy years ago the standards for mathematics differed from those of today, and despite the fact that students would receive an outstanding education from a teacher who could present the information in this fashion, the current standards would be more difficult to meet. It is noble of any teacher to have the desire and drive to attempt such an endeavor, with the completeness of Fawcett's study.

I do think that teachers can pull important techniques and activities from the study in an effort to put both teaching and learning into the hands of the students. The most beneficial part of Fawcett's class was that the students depended on resources that they usually use sparingly: namely <u>their own experiences and knowledge and the ideas, thoughts, and conclusions of their peers</u>. So often students sit and listen to a teacher lecture day in and day out and work most of their problems out of a previously written textbook. If students feel that they have ownership in their learning, many of them will rise to the occasion without knowing that they are becoming directly involved their own education. In traditional classrooms, how often do students get to lead a discussion or disagree with an assumption or proof on the board? According to this book, they need more of this time in order to feel passionate about the discipline.

The ability to recall geometry in the experimental group, although different from the traditional class, was enhanced through the creation of their own geometry. Even though these students did not memorize axioms or proofs, they are more likely to remember the methods that they used to determine certain aspects of the geometry and can apply these techniques to future mathematics classes. If a problem is presented that the traditional class only memorized, this will probably not remain in their long-term memory. The experimental class, on the other hand, could take the methods they used to prove such a problem and work through it and various other problems. Since they were using these methods in a variety of ways throughout life, these techniques were less likely to be lost compared to the memorized proof.

One fundamental aspect of mathematics is communication. The students in the experimental group had to establish top communication skills in order to explain their ideas, rationale, and proofs to the other students. They were forced (without knowing) to learn the vocabulary of a mathematician and use it everyday. Without the proper communication skills, other students had the opportunity to disagree with their peers. In high school, teenagers find it essential to be accepted by their peers, and this encouraged being prepared for classroom discussions. Outside of the mathematics portion of the class, students also worked with governmental, school, and various other topics in which communication through reading and explanation was necessary. In the end, students had created their own textbooks. What a great way to communicate with other mathematicians!

Harold Fawcett created an "ideal" classroom in which students had the freedom to explore geometry and methods of proof at their own rate, collaborate information in groups and as a class, and learn through teaching, not lecture. Seventy-two years later, teachers of geometry in the public high schools have not attempted this method or even taken his main ideas in an effort to give more meaning to a child's education. Recreating the entire study is an unrealistic goal, but his approaches to teaching in general can be carried to all disciplines. After reading *The Nature of Proof*, I encourage both mathematicians and mathematics educators to take the pieces of Fawcett's study that will benefit each individual classroom to open the minds of the students. If teachers give students reasons for learning and an ownership in their education, the students will retain mathematical processes and become more engaged and interested in the study of mathematics. This may come across as an idealistic vision but as John Lennon once said: "You may say I'm a dreamer, but I'm not the only one…"

For more information regarding:

- No Child Left Behind visit http://www.ed.gov/nclb/landing.jhtml
- Montana Standardized Tests 2002-2003 visit http://www.opi.state.mt.us/PDF/Superintendent/HowToInterpretScores.pdf
- The Nation's Report Card visit http://nces.ed.gov/nationsreportcard/sitemap.asp