Math 422: Abstract Algebra II
Homework 6, Due Friday, March 14

- Do the following problems:
  - Chapter 11: # 32.
- Reduce to normal form the matrix

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

- Suppose G is the quotient of $\mathbb{Z}^{\oplus 3}$ by the subgroup of all linear combinations of the elements $3e_1 + 2e_2 + e_3$, $8e_1 + 4e_2 + 2e_3$ and $7e_1 + 6e_2 + 2e_3$. Find a free group H and a direct sum of finite cyclic groups T such that $G \cong H \oplus T$.
- In this exercise (which is in Munkres, Elements of Algebraic Topology) you will prove the uniqueness part of the structure theorem for finitely generated abelian groups.
  1. Let p be prime; let $b_1, \ldots, b_k$ be non-negative integers. Show that if

$$G \cong (\mathbb{Z}_p)^{\oplus b_1} \oplus (\mathbb{Z}_{p^2})^{\oplus b_2} \oplus \cdots \oplus (\mathbb{Z}_{p^k})^{\oplus b_k}$$

     then the integers b_i are uniquely determined by G. [Hint: Consider the kernel of the homomorphism $f_i : G \to G$ that is multiplication by p^i. Show that f_1 and f_2 determine b_1. Proceed similarly.]
  2. Let $p_1, \ldots, p_N$ be a sequence of distinct primes. Generalize the first part to a finite direct sum of terms of the form $(\mathbb{Z}_{p_i{}^k})^{\oplus b_{ik}}$ where $b_{ik} \geq 0$.
  3. Verify the third part of the structure theorem.